B-III Analysis IV Final examination 2017.

Answer all the 10 questions. Each question is worth 6 points.

If you are using any result proved in the class, you need to state it correctly. If the answer is an **immediate** consequence of the result quoted by you, you need to **prove** the result quoted by you. We consider R^n equipped with Euclidean metric and length function.

1. Show that the set of irrational numbers with the usual metric is a separable metric space.
2. Let (X, d) be a complete metric space. Let $A \subset X$ be a totally bounded set. Show that $\overline{A}$ is a compact set.
3. Let (X, d) be a separable metric space. Let $\{U_\alpha\}_{\alpha \in \Delta}$ be a family of open sets such that $X = \cup_{\alpha \in \Delta} U_\alpha$. Give detailed proof to show that there is a countable set $A \subset \Delta$ such that $X = \cup_{\alpha \in A} U_\alpha$.
4. Consider the space $\ell^2 = \{ \{\alpha_n\}_{n \geq 1} : \sum_1^\infty |\alpha_n|^2 < \infty \}$, with the usual metric. Let $\mathcal{F} = \{ \{\alpha_n\} \in \ell^2 : \sum_1^\infty |\alpha_n|^2 \leq 1 \}$. Show that $\mathcal{F}$ is not a compact set.
5. Let $f : R \to R^3$ be such that f is differentiable and $f(R) \subset \{x \in R^3 : \|x\| = 1\}$. Show that for every $t \in R$, $f(t)$ is orthogonal to $f'(t)$.
6. Let $f : R^2 \to R^2$ be a continuously differentiable function. Suppose $f'(0)$ has non-zero determinant. Let $U = \{ x \in R^2 : \|f'(x) - f'(0)\| < \frac{1}{2\|f'(0)\|} \}$. Show that $f(U)$ is an open set.
7. Let $f : [1, \infty) \to R$ be such that f is Riemann integrable in $[1, a]$ for all $a \geq 1$ and $\int_1^\infty f(t)dt$ converges absolutely. Show that
$$\lim_{\alpha \to +\infty} \int_1^\infty f(t) sin\alpha t \; dt = 0.$$
8. Let g be a continuous function of bounded variation on $[0, 1]$ such that $g(1) = 1$. Derive the formula,
$$\frac{\pi}{2} = \lim_{\alpha \to +\infty} \int_0^1 g(t) \frac{sin\alpha t}{t} \; dt.$$

9. Let $\{\phi_n\}_{n\geq 0}$ be an orthonormal sequence on $[0,\pi]$. Let $\mathcal{R}$ denote the set of all Riemann integrable functions on $[0,\pi]$. For a $f \in \mathcal{R}$, let $a_n = \int_0^\pi f(t)\overline{\phi_n(t)}dt$. Suppose for every $f \in \mathcal{R}$, $\int_0^\pi |f|^2 dt = \sum_0^\infty |a_n|^2$. For $f, g \in \mathcal{R}$, let $\{a_n\}_{n\geq 0}$, $\{b_n\}_{n\geq 0}$ are the corresponding sequences of Fourier coefficients. Show that the series $\sum_0^\infty a_n\overline{b_n}$ converges and

$$\sum_0^\infty a_n\overline{b_n} = < f, g > .$$

10. Let $f : [a,b] \to R$ be a Riemann integrable function. Let $\{\psi_n\}_{n\geq 1}$ be an orthonormal sequence on $[a,b]$. Suppose the Cesaro sums corresponding to the Fourier series of f converge uniformly to f. Show that

$$\int_a^b |f(t)|^2 dt = \sum_1^\infty |\int_a^b f(t)\overline{\psi_n(t)}\, dt|^2.$$