Indian Statistical Institute, Bangalore
B. Math. Third Year
Second Semester - Analysis IV

Semestral Exam | Duration: 3 hours | Date : April 28, 2015

Section I: Answer any four, each question carries 6 marks

1. If X is a compact metric space, prove that $C(X)$ is a separable metric space.

2. If X is a compact metric space and $\mathcal{A}$ is a closed subalgebra of $C_{\mathbb{R}}(X)$ that separates points of X, prove that $\mathcal{A} = C_{\mathbb{R}}(X)$ or there is a $x_0 \in X$ such that $\mathcal{A} = \{f \in C_{\mathbb{R}}(X) \mid f(x_0) = 0\}$.

3. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be $f(x, y) = (x^2 - y^2, 2xy)$. Prove that f is locally one-one but not one-one on $\mathbb{R}^2 \setminus (0, 0)$ and discuss inverse function theorem at $(1, 1)$.

4. Let $f \in \mathcal{R}[-\pi, \pi]$ be a 2π-periodic function and $s_n(x)$ be the n-th partial sum of the Fourier series at $x \in \mathbb{R}$. Prove that for $x \in \mathbb{R}$,
$$\frac{1}{n}\sum_{i=0}^{n-1} s_i(x) = \frac{1}{2n\pi}\int_{-\pi}^{\pi} \frac{f(x+t) + f(x-t)}{2} \frac{\sin^2 \frac{nt}{2}}{\sin^2 \frac{t}{2}} dt.$$

5. Prove that $\sum_1^\infty \frac{\sin(2n-1)x}{2n-1} = \frac{\pi}{4}$ for $0 < x < \pi$.

6. Let $f(x) = 1$ if $|x| \leq 1$, $f(x) = 0$ if $1 < |x| \leq \pi$ and $f(x + 2\pi) = f(x)$ for all $x \in \mathbb{R}$. Find the Fourier coefficients of f and deduce that $\sum_1^\infty \frac{\sin n}{n} = \frac{\pi - 1}{2}$.

Section II: Answer any two, each question carries 13 marks

1. (a) Show that the set of all polynomials of degree at most 3 with coefficients from $[-1, 1]$ is compact in $C[0, 1]$. Does the result hold if coefficients are not assumed to be from $[-1, 1]$ (**Marks: 7**).

   (b) Prove that $\Omega = \{A \in L(\mathbb{R}^n) \mid \det(A) \neq 0\}$ is open and $A \mapsto A^{-1}$ is continuous on Ω.

2. (a) Let $E \subset \mathbb{R}^{n+m}$ be an open set and $f : E \to \mathbb{R}^n$ be a C^1-map. Assume $f(a, b) = 0$ and A_x is invertible where $A = f'(a, b)$. Prove that there is neighborhood U of (a, b) such that $\{(f(x, y), y) \mid (x, y) \in U\}$ is open.

   (b) Prove that for $0 < x < 2\pi$, $x^2 = \frac{4}{3}\pi^2 + 4\sum_1^\infty [\frac{\cos nx}{n^2} - \frac{\pi \sin nx}{n}]$ (**Marks: 6**).

3. Let $f \in \mathcal{R}[-\pi, \pi]$ be a 2π-periodic function and $s_n(x)$ be the n-th partial sum of the Fourier series at $x \in \mathbb{R}$.

(a) If $s(x) = \lim_{t \to 0} \frac{f(x+t)+f(x-t)}{2}$ exists for some $x \in [-\pi, \pi]$, prove that $\sigma_n(x) = \frac{1}{n} \sum_{i=0}^{n-1} s_i(x) \to s(x)$.

(b) If f is differentiable such that $f' \in \mathcal{R}[-\pi, \pi]$ and $\frac{1}{2\pi} \int_{-\pi}^{\pi} |f'(t)|^2 dt \leq 1$. Prove that $|f(x) - s_n(x)| \leq \frac{2}{\sqrt{n}}$ for all $x \in \mathbb{R}$ and $n \geq 1$ (**Marks: 7**).