Indian Statistical Institute
II Semestral Examination 2008-2009
B.Math.(Hons). II year
Analysis IV

Date:04-05-2009 Duration: 3 Hours Instructor: B.Bagchi

Max Marks 100

1. For $k \varepsilon Z$, let $\varepsilon_k \varepsilon L^2([0, 2\pi])$ be given by $\varepsilon_k(x) = e^{ix}$. For $f \varepsilon L^2$, let $a_k = (f, \varepsilon_k)$ be the Fourier coefficients of f and let b_k be an arbitrary complex number ($k \varepsilon Z$).

   a) Show that $\|f - \sum_{k=-n}^{n} a_k \, \varepsilon_k\| \leq \|f - \sum_{k=-n}^{n} b_k \, \varepsilon_k\|$ for every $n \geq 0$.

   b) Show that equality holds here for some n only if $b_k = a_k$ for $-n \leq k \leq n$. [15 + 5 = 20]

2. a) Define the normed linear space $L^1([0, 2\pi])$ carefully and show that it is a Banach space.

   b) Show that it is not a Hilbert space, *i.e.* its norm does not "come" from any inner product. [15 + 10 = 25]

3. Let $f : \mathbb{R} \longrightarrow \mathbb{C}$ be a continuous function which is periodic with period 2π.

   a) Show that the Fourier series of f converges to f in the sense of Cesaro, uniformly on compact subsets of $\mathbb{R}$.

   b) If, further, f has continuous derivative everywhere, then show that the Fourier series of f converges to f uniformly on compact subsets of $\mathbb{R}$, in the usual sense. [15 + 15 = 30]

4. Let X be a metric space with $\sharp\,(X) \geq 5$.

   a) If every five point subset of X embeds isometrically in $\mathbb{R}^2$ (with euclidian metric) then show that X embeds in $\mathbb{R}^2$.

   b) Give an example of a metric space X with $\sharp\,(X) = 5$ such the every proper subset of X embeds in $\mathbb{R}^2$ but X does not embed in $\mathbb{R}^2$.

   [10 + 15 = 25]