SEM 1, 2023-24: ALGEBRAIC TOPOLOGY
MID-SEMESTRAL EXAMINATION

Each question carries 5 marks. Total: 30 marks. Time: 3 hours.
You may use $\pi_1(\mathbb{S}^1) \simeq \mathbb{Z}$ without proof. You may quote any result proved in class without proof.

(1) (a) Show that the fundamental group of any path-connected subspace of $\mathbb{R}^2$ containing the unit circle and not containing the origin contains a copy of $\mathbb{Z}$.
(b) Show that every map from $\mathbb{R}P^2 \to \mathbb{S}^1$ is nullhomotopic.

(2) How many connected 2-sheeted covers $p : Y \to X := \mathbb{S}^1 \vee \mathbb{S}^1$ of X are there up to isomorphism? Justify. Conclude that the deck transformation group $Deck(p)$ acts transitively on the fibers of p for all these coverings. Give a presentation of $\pi_1(Y)$ in terms of generators and relations for all Y (in terms of generators of X).

(3) (a) Show that cone of any topological space is contractible.
(b) Use this to show that the fundamental group of suspension of any path-connected space X is trivial.
(c) Give a counter-example when X is not path-connected.

(4) Construct a topological space whose fundamental group is $\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ (for any two positive integers m, n).

(5) (a) Recall that the Möbius strip M is the quotient space of the square $[0,1] \times [0,1]$ obtained by identifying $(0,t)$ with $(1, 1-t)$ for all $t \in [0,1]$. Compute the fundamental group of M.
(b) Prove that there is no retraction of M to its boundary, i.e., there is no continuous map $r : M \to \partial M \simeq \mathbb{S}^1$ such that $r \circ i = id_{\partial M}$, where $i : \partial M \to M$ denotes the inclusion.
(c) If you glue two copies of the Möbius strip along their boundaries, you get a surface K. Use Seifert–van Kampen theorem to find a presentation for the fundamental group of this surface.

(6) Suppose $f_t : X \to X$ is a homotopy such that f_0 and f_1 are each the identity map. For any $x_0 \in X$, show that the class of the loop $t \mapsto f_t(x_0)$ is in the center of $\pi_1(X, x_0)$.

Date: 22-09-2023.