February 24th, 2023
TDS - Midterm - Semester II

Instructions: Closed Book Exam. Please write detailed and correct answers.

1. Let $\alpha > 0$. Consider the graph $\mathbb{Z}_+$ with weights $\mu^{\alpha}_{n,n+1} = \alpha^n$.
   (a) (5 points) Show the graph $(\mathbb{Z}_+, \mu^{\alpha})$ has controlled weights. Does it have bounded weights ?
   (b) (10 points)Show that the graph is recurrent if and only if $\alpha \leq 1$.

2. Let X_n be a random walk on $\mathbb{Z}^3$ with natural weights. Let T_0 be the hitting time of 0. Let $n \geq 1$, $A = \mathbb{Z}^3 \setminus \{0\}$. Let $h_n, h : \mathbb{Z}^3 \to [0,1]$ be given by

$$h_n(x) = \mathbb{P}^x(T_0 \geq n) = \mathbb{P}^x(X_k \in A, 1 \leq k \leq n)$$

and

$$h(x) = \mathbb{P}^x(T_0 = \infty) = \mathbb{P}^x(X_n \in A, \text{ for all } n \geq 0).$$

   (a) (8 points) Show that $h_n = Q^n 1_A$ and $h = Qh$, where Q is the restriction of P onto A.
   (b) (2 points) Suppose $\alpha = \sup_{x \in A} h(x)$, show that $0 < \alpha \leq 1$ and $h \leq \alpha 1_A$
   (c) (3 points) Using (i) and (ii), conclude that $h \leq \alpha h_n$
   (d) (2 points) Conclude that $\max_{x \in \partial A} h(x) \neq \sup_{x \in A} h(x)$.

3. Consider Γ to be the join of two copies of $\mathbb{Z}^3$ at their origins. Write $\mathbb{Z}^3_{(i)}, i = 1,2$ the two copies, and 0_i for their origins. Let

$$F = \{X \text{ is ultimately in } Z^3_{(1)}\}$$

and let $h(x) = \mathbb{P}^x(F)$.
   (a) (3 points) Show that h is harmonic,
   (b) (4 points) Show that $h(x) \geq \mathbb{P}^x(X \text{ never hits } 0_1)$ for $x \in \mathbb{Z}^3_{(1)}$.
   (c) (4 points) Show that $h(x) \leq \mathbb{P}^x(X \text{ hits } 0_2)$ for $x \in \mathbb{Z}^3_{(2)}$.
   (d) (4 points) Decide whether Γ has the Liouville Propety: All bounded harmonic functions on Γ are constant.

4. Let $H^2(V) = \{f \in C(V) : \mathcal{E}(f,f) < \infty\}$ where

$$\mathcal{E}(f,g) = \frac{1}{2}\sum_{x\in V}\sum_{y\in V} \mu_{xy}(f(x) - f(y))(g(x) - g(y)) \text{ and } \| f \|_{H^2} = \mathcal{E}(f,f) + f(\rho)^2,$$

for $f, g \in C(V)$ and (fixed) $\rho \in V$. Show that $H^2(V)$ is a Hilbert space.