MID-SEMESTER EXAMINATION, COMPLEX ANALYSIS, 2013-14

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A1. Let $z_0 \in U$. Then $f(z_0) \neq 0$. Choose an open ball $B(z_0)$ around z_0 which does not contain the origin. Let $U_1 = f^{-1}(B(z_0))$. Therefore U_1 is an open set containing z_0

Again, as $B(z_0)$ is a simply connected open set not containing 0, so $z^{1/2}$ is a holomorphic function on it. This implies that on U_1, $f = z^{1/2} \circ f^2$ is holomorphic. So, f is holomorphic on U.

Let $g(z) = z^{1/2}$. On U_1, $f'(z_0) = g'(f^2(z_0)).((f^2(z))'(z_0)) = \frac{1}{2f(z_0)}.((f^2(z))'(z_0))$.

A2. We shall prove that the image of an entire function is dense in $\mathbb{C}$. This result will directly imply that in both cases f is constant.

Let, if possible, let the image of f is not dense. Then there is an open ball $B(z_0, r)$ around z_0 which is not contained in the image. Then let $g = f - z_0$ and $B(0, r)$ is not in the image of g. Now, as $\frac{1}{z}$ is a holomorphic function on $\mathbb{C} - 0$, so $h(z) = 1/z \circ g$ is an entire function with $\mid h(z) \mid \leq 1/r$. So, $h(z)$ is a bounded entire function, so is a constant function. So, g is also a constant function and so is f.

A3. If possible, suppose that there is no $c > 0$ such that

$$\sup(|1/z - p(z)| \colon |z| = 1) > c.$$

for all $p(z) \in [z]$.
Then there is a sequence $p_n(z) \in \mathbb{C}[z]$ with $|1/z - p_n(z)| < 1/n$ for all $|z| = 1$. Therefor, $p_n(z)$ converges uniformly to $1/z$ on unit circle S^1. Then $\int_{S^1} p_n(z)dz$ converges to $\int_{S^1}(1/z)dz$. By Cauchy's integral formula, $\int_{S^1} p_n(z)dz = 0$ and $\int_{S^1}(1/z)dz = 2\pi i$, which is a contradiction.

A4. The set $\mathbb{H}$ is a simply connected open subset of $\mathbb{C}$, not containing -1. Then $\frac{1}{1+z}$ is a holomorphic function on $\mathbb{H}$. Now, if we take the curve γ_1, which is γ_2 followed by γ_3, defined by
$\gamma_2(t)$ is the straight line from $-1 + i$ to $-1 + 2i$,
$\gamma_3(t)$ is the straight line from $-1 + 2i$ to $1 + 2i$.
As in $\mathbb{H}$, γ and γ_1 are homotopic and $\frac{1}{1+z}$ is holomorphic on $\mathbb{H}$, so $\int_\gamma \frac{1}{1+z} = \int_{\gamma_1} \frac{1}{1+z}$.
Now, $\int_{\gamma_1} \frac{1}{1+z} = \int_{\gamma_2} \frac{1}{1+z} + \int_{\gamma_3} \frac{1}{1+z}$.
Let $z = -1 + it$, then
$\int_{\gamma_2} \frac{1}{1+z} = \int_1^2 \frac{idt}{it} = \log 2 - \log 1 = \log 2$.
Let $z = t + 2i$.
Then $\int_{\gamma_3} \frac{1}{1+z} = \int_{-1}^1 \frac{dt}{1+2i+t} = \int_{-1}^1 \frac{1-2i+t}{(1+t)^2+4}dt = (1-2i)\int_{-1}^1 \frac{dt}{(1+t)^2+4} + \int_{-1}^1 \frac{tdt}{(1+t)^2+4}$
$= (1-2i)\int_0^2 \frac{dt}{t^2+4} + \int_0^2 \frac{tdt}{t+4} = (1-2i)\frac{1}{2}.\frac{\pi}{4} + \frac{1}{2}(\log 8 - \log 4)$.

A5. As f is a holomorphic function, so we have $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$. Now, as u is a function of x and v is a function of y, so $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = c$, where c is a complex number. So, $u = cx + d_1$ and $v = cy + d_2$, where $d_1, d_2 \in \mathbb{C}$. Therefore, $f = cz + d_1 + id_2$. Hence, f is a polynomial with degree ≤ 1.

A6. (i) Suppose that for all $z \in \mathbb{D}$, $f(z) \neq 0$. Again by Maximum Modulus Principle, $|f(z)|< 1 \forall z \in \mathbb{D}$. But, if we can take the holomorphic function $\frac{1}{z} \circ f$, then we have $|\frac{1}{z} \circ f(z)|> 1\ \forall z \in \mathbb{D}$ and $|\frac{1}{z} \circ f(z)|= 1\ \forall |z|= 1$, which gives a contradiction to Maximum Modulus Principle. So, there will be $z \in \mathbb{D}$, so that $f(z) = 0$.

(ii) $|\phi_\alpha(z)|^2= \phi_\alpha(z)\overline{\phi_\alpha(z)} = \frac{z-\alpha}{1-\bar{\alpha}z}.\frac{\bar{z}-\bar{\alpha}}{1-\alpha\bar{z}} = \frac{z\bar{z}-\alpha\bar{z}-\bar{\alpha}z+\alpha\bar{\alpha}}{1-\bar{\alpha}z-\alpha\bar{z}+\alpha\bar{\alpha}z\bar{z}}$.
So, $|z|= 1$ implies $|\phi_\alpha(z)|= 1$.
(iii) Take $g = \phi_\alpha \circ f$. Then from (i), for some $z \in \mathbb{D}$, $g(z) = 0$, i.e. $\frac{f(z)-\alpha}{1-\bar{\alpha}f(z)} = 0$. So for some $z \in \mathbb{D}$, we have $f(z) = \alpha$.

A7. Take $r < 1$ and let γ be the closed curve $\gamma(t) = re^{2\pi it}$. Then $a_n = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z^{n+1}}dz$, where $f(z) = \sum a_n z^n$ is a holomorphic function on $\mathbb{D}$. Then, $|a_n|\leq \frac{1}{2\pi}\int_\gamma \frac{|f(z)|}{r^{n+1}} \leq \frac{1}{r^{n+1}}$ and this is true for any $0 < r < 1$, therefore $|a_n|\leq 1$.