Your Signature ______________________

Instructions:
1. For writing your answers use both sides of the paper in the answer booklet.

2. Please write your name on every page of this booklet and every additional sheet taken.

3. If you are using a Theorem/Result from class please state and verify the hypotheses of the same.

4. **Maximum time is 2 hours and Maximum Possible Score is 100.**

Score

Q.No.	**Alloted Score**	**Score**
1.	17	
2.	17	
3.	17	
4.	17	
5.	17	
6.	17	
Total	102	

Number of Extra sheets attached to the answer script: ______________________

1. Let $\{Y_n\}_{n\geq 1}$ be a sequence of bounded random variables on $(\Omega, \mathcal{F}, \mathbb{P})$. Show that

$$\bar{Y} = \limsup_{n\to\infty} Y_n$$

is measurable.

2. Let $\{X_n\}_{n\geq 1}$ be independent random variables on $(\Omega, \mathcal{F}, \mathbb{P})$. Suppose $X_n \sim \text{Poisson}(1)$ then show that

$$\mathbb{P}(\limsup_{n\to\infty} X_n \frac{\log\log(n)}{\log(n)} = 1) = 1.$$

3. Let $\{X_n\}_{n\geq 1}$ be non-negative i.i.d. random variables.

   (a) Suppose $\lim_{n\to\infty} \frac{X_1+X_2+\ldots+X_n}{n} = c \in \mathbb{R}$ a.s. then is $c = E[X]$?

   (b) Suppose $E[X] = \infty$ then can $\limsup_{n\to\infty} \frac{X_1+X_2+\ldots+X_n}{n} \in \mathbb{R}$ with positive probability?

4. Let Z_n be i.i.d random variables on $(\Omega, \mathcal{F}, \mathbb{P})$such that

$$\mathbb{P}(Z_n = 1) = \frac{1}{2} = 1 - \mathbb{P}(Z_n = -1).$$

Define $X_n = \frac{Z_n}{n^\theta}$ for $0 < \theta$. Decide whether the series with partial sums $S_n = \sum_{j=1}^n X_n$ converges almost surely or not ?

5. Let $\mathbb{P}, \{\mathbb{P}_n\}_{n\geq 1}$ be Probability measures on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$. Suppose that for every subsequence $\mathbb{P}_{n_k}$ there is a further subsequence $\mathbb{P}_{n_{k_l}}$ that converges weakly to $\mathbb{P}$. Show that $\mathbb{P}_n$ converge weakly to $\mathbb{P}$.

6. Let $\{A_n\}_{n\geq 1}$ be a sequence of pairwise independent events. Fix $n \geq 1$ and let $X_m = \sum_{i=n}^{m} 1_{A_i}$ for $m > n$.

(a) Show that $P(X_m \geq 1) \geq \frac{1}{1+(\sum_{k=n}^{m} P(A_k))^{-1}}$

(b) Using (a) show that if $\sum_{k=1}^{\infty} P(A_k) = \infty$ then $P(A_n \text{ occur i.o.}) = 1$