Indian Statistical Institute

Final Exam. 2023-2024

Measure Theory, M.Math First Year

Time : 3 Hours   Date : 15.11.2023   Maximum Marks : 100   Instructor : Jaydeb Sarkar

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(i) Answer all questions. (ii) You are allowed to use any of the theorems that were covered in class. (iii) μ represents a positive measure. (iv) $m =$ the Lebesgue measure on $\mathbb{R}$.

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(1) (15 marks) Prove or disprove: If f is continuous and has a bounded variation on $[0,1]$, then f is absolutely continuous on $[0,1]$.

(2) (15 marks) Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $f \in L^1(\mu)$. Suppose

$$\Big| \int_X f \, d\mu \Big| = \int_X |f| \, d\mu.$$

Prove that either $f \geq 0$ or $f \leq 0$ a.e. on X.

(3) (15 marks) Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $f \in L^1(\mu)$. Suppose

$$0 \leq f(x) \leq 1 \qquad (x \in X).$$

Prove that

$$\lim_{n\to\infty} \int_X f^n \, d\mu = \mu(f^{-1}(\{1\})).$$

(4) (15 marks) Let A be a Lebesgue measurable subset of $\mathbb{R}^2$. Suppose

$$m(\{x : (x,y) \in A\}) = 0,$$

for all y in $\mathbb{R}$ a.e. Prove that the Lebesgue measure of A is zero. Also prove that

$$m(\{y : (x,y) \in A\}) = 0,$$

for all x in $\mathbb{R}$ a.e.

(5) (15 marks) Let $f_1, f_2 \in L^1([0,1])$. Suppose $f_1(x), f_2(x) > 0$ for all $x \in [0,1]$. Define

$$\nu_i(E) = \int_E f_i \, dm \qquad (i = 1, 2),$$

for all Lebesgue measurable subset $E \subseteq [0,1]$. Prove that $\nu_1 << \nu_2$. Also compute $\frac{d\nu_1}{d\nu_2}$.

(6) (15 marks) A normed linear space X is said to be separable if there is a countable dense subset of X. Prove that $L^\infty(\mathbb{R}^n)$ is not separable. Here, $\mathbb{R}^n$ is equipped with the Lebesgue measure.

(7) (4+4+12 = 20 marks) Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $1 \leq p < \infty$. Suppose $\{f_n\}_n \subseteq L^p(\mu)$.

(i) If

$$\sum_n \|f_n\|_p < \infty,$$

then prove that there exists $f \in L^p(\mu)$ such that

$$f(x) = \sum_n f_n(x) \quad (x \in X \text{ a.e.})$$

and

$$f = \sum_n f_n,$$

in $L^p(\mu)$.
(ii) Prove that if $f_n \to g$ in $L^p(\mu)$, then $\{f_n\}_n$ has a subsequence which converges pointwise a.e. to g.
(iii) Prove that $L^\infty(\mu) \cap L^p(\mu)$ is a Borel subset of $L^p(\mu)$.