Indian Statistical Institute, Bangalore
M. Math First Year
First Semester - Measure Theory

Semestral Exam | Date: January 07, 2022
Maximum marks: 60 | Duration: 3 hours

Answer all questions and all questions carry equal marks.

1. Let X be an uncountable set and $\mathcal{A}$ be the collection of subsets A of X such that A or $X \setminus A$ is countable. Prove that $\mathcal{A}$ is a σ-algebra and the smallest σ-algebra containing the singletons. Determine all measurable functions on X.

2. Let A and B be two subsets of $\mathbb{R}$ such that $\inf\{|a-b| \mid a \in A, b \in B\} > 0$. Prove that $m^*(A \cup B) = m^*(A) + m^*(B)$ where m^* is the Lebesgue outer measure on $\mathbb{R}$.

3. Let $(X, \mathcal{A}, \mu)$ be a measure space and $\mu(X) = 1$. Prove that for $1 \leq r < s \leq \infty$, $L^s \subset L^r$.

4. Let m be the Lebesgue measure on $\mathbb{R}$. For a Lebesgue integrable function f on $\mathbb{R}$, prove that
$$m \times m(\{(x, y) \mid 0 \leq y \leq f(x)\}) = \int f(x) dm(x) = \int_0^\infty \phi(t) dm(t)$$
where $\phi(t) = m(\{x \mid f(x) \geq t\}$.

5. Let $\nu_j \ll \mu_j$ be σ-finite measures on $(X_j, \mathcal{A}_j)$ for $j = 1, 2$. Then $\nu_1 \times \nu_2 \ll \mu_1 \times \mu_2$ and for a.e. $(x, y) \in X_1 \times X_2$, $\frac{d(\nu_1 \times \nu_2)}{d(\mu_1 \times \mu_2)}(x, y) = \frac{d\nu_1}{d\mu_1}(x) \frac{d\nu_2}{d\mu_2}(y)$.