Indian Statistical Institute, Bangalore Centre.
Mid-Semester Exam : Markov Chains (B3 and M2)

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Max. points : 40. **Time Limit : 3 hours.**

There are two parts to the question paper - PART A and PART B. Read the instructions for each section carefully. Some common notations, definitions and a useful result are listed on Page 3.

1 PART A : MULTIPLE-CHOICE QUESTIONS - 10 Points.

Please write only the correct choice(s) (for ex., (a), (b) et al.) in your answer scripts. No explanations are needed. Write PART A answers in a separate page.

Some questions will have multiple correct choices. Answer all questions. Each question carries 2 points. 1 point will be awarded if only some correct choices are chosen and no wrong choices are chosen.

1. Consider simple lazy random walk on the cycle graph C_n on n vertices for $n \geq 3$. Which of the following is true for all $n \geq 3$?
   (a) It is irreducible.
   (b) It is aperiodic.
   (c) It has a unique stationary distribution.
   (d) $P(x, y) > 0$ for all $x, y \in C_n$?
2. Consider simple random walk on the complete graph K_n on n vertices for $n \geq 3$. Which of the following is true for all $n \geq 3$?
   (a) It is irreducible.
   (b) It is periodic.
   (c) Uniform distribution is the stationary distribution.
   (d) $P(x, y) > 0$ for all $x, y \in K_n$?
3. Let P be a transition matrix. Which of the following are transition matrices ?
   (a) $Q = P/2$.
   (b) $Q = P + I$, where I is the identity matrix.

(c) $Q = P^{2511}$.
(d) $Q = \frac{P}{4} + \frac{3I}{4}$

4. Let P be a transition matrix reversible w.r.t. probability distribution π. Which of the following are true ?

   (a) π is stationary distribution for P.
   (b) $\frac{P+I}{2}$ is reversible w.r.t. π.
   (c) P^2 is not a reversible transition matrix.
   (d) $\frac{P+I}{2}$ does not have a stationary distribution.

5. Consider the biased random walk on $\mathbb{Z}$ with parameters $p, q = 1-p > 0$. Which of the following are true ?

   (a) It is irreducible.
   (b) $P^{2023}(0,0) > 0$.
   (c) $P^{2022}(0,2022) > 0$.
   (d) $P^2(0,2) = P^2(2,0)$.

2 PART B : 30 Points.

Answer any three questions only. All questions carry 10 points.

Give necessary justifications and explanations for all your arguments. If you are citing results from the class, mention it clearly. Always define the underlying random variables and events clearly before computing anything !

1. For an HMC $\{X_n\}$ on $\mathbb{Z}$, let $f_{i,j} = \mathbb{P}_i(\tau_j < \infty), i, j \in \mathbb{Z}$ where τ_j is the hitting time to j. Let N_j denote the number of visits by $\{X_n\}$ to j. Show that the following hold :

   (a) For $m \geq 1$, $\mathbb{P}_j(N_j = m) = f_{j,j}^{m-1}(1 - f_{j,j})$ **(5)**
   (b) For $i \neq j$, $\mathbb{P}_i(N_j = 0) = 1 - f_{i,j}$; $\mathbb{P}_i(N_j = m) = f_{i,j} f_{j,j}^{m-1}(1 - f_{j,j}),\ m \geq 1$. **(5)**

2. The complaint register system at an institute works as follows. Every day, one of the following two events happen - A new complaint arrives with probability p or an existing complaint (if there is any) is attended to with probability $q = 1 - p$ (i.e., if there is no existing complaint, nothing happens with probability q). Let X_t be the number of unattended complaints in the institute at the end of day t. Show that X_t is positively recurrent if $q > p$ and find the stationary distribution for the Markov chain.

3. Let $G_d = (\mathbb{Z}^d, E_d)$ be the d-dimensional lattice with the usual edge-set E_d, $d \geq 1$. Define $G'_d = (\mathbb{Z}^d, E'_d)$ where $E'_d = E_d \bigcup \cup_{i \neq j \in \{1,\dots,d\}} \{(e_i, e_j)\}$ where e_i is the ith co-ordinate basis vector in $\mathbb{R}^d$. Denoting $\mathcal{R}_d, \mathcal{R'}_d$ to be effective resistances in G_d, G'_d respectively and $O = (0, \dots, 0)$ to be the origin, show the following.

   (a) $\mathcal{R}_d(O \leftrightarrow \infty) \geq \mathcal{R}'_{d+1}(O \leftrightarrow \infty)$. **(6)**
   (b) $\mathbb{P}'_O(\tau_O^+ = \infty) \geq \mathbb{P}_O(\tau_O^+ = \infty)$, where $\mathbb{P}, \mathbb{P}'$ denote the probabilities in G_d, G'_d respectively. **(4)**

4. Consider the lazy random walk on the hypercube $\Omega_n := \{0,1\}^n$ and let π be its stationary distribution. Let $f : \Omega_n \to \mathbb{R}$ be defined as $f(x) = \sum_{i=1}^n x_i, x = (x_1, \ldots, x_n)$. Show the following.

   (a) Find a bound for t such that $\max_{x\in\Omega_n} |\mathbb{P}_x(f(X_t) = k) - \binom{n}{k}2^{-n}| \leq 1/4$ for all k. (**5**)

   (b) $\mathbb{V}AR_\pi(P^t f) \leq \frac{n}{4}(\frac{n-1}{n})^{2t}$ (**5**)

5. Let $X_t, t \geq 0$ be an HMC on a finite set Ω. Fix $k \geq 1$

   (a) Show that $Y_t = (X_t, \ldots, X_{t+k}), t \geq 0$ is an HMC and finds its transition matrix. (**4**)

   (b) If X_t is irreducible, is Y_t irreducible ? (**3**)

   (c) If π is the stationary distribution for X_t, find a stationary distribution for Y_t. (**3**)

Some notations :

- HMC - Homogeneous Markov chain. P - Transition matrix of HMC.
- A random walk on a network (weighted graph) $G = (V, E, c)$ is the Markov chain with transition matrix $P(x, y) = \frac{c(x,y)}{c(x)}$ where $c(x) = \sum_{y:y\sim x} c(x, y)$. Here $c : E \to [0, \infty)$.
- If no weights are specified, then $c(x, y) = 1[x \sim y]$.
- A lazy random walk is the Markov chain with transition matrix $\frac{P+I}{2}$ with P as above.
- Biased random walk on $\mathbb{Z}$: $P(x, x + 1) = p = 1 - P(x, x - 1)$ for all $x \in \mathbb{Z}$.
- $t_{mix} \leq \log(\frac{4}{\min_x \pi(x)})t_{rel}$.