MMath-I, Linear Algebra Final (Back paper)

Instructions: Total time 3 Hours. All questions carry equal weightage.

1. Give an example of a matrix $A = (a_{ij})$ of size 5×5 with complex entries such that $A^2 = 0$ and all entries of A are nonzero.

2. **(i)** Let k be a field. Let $V = k^n$ with elements written as row vectors and $W = k^m$ with elements written as column vectors. Prove that $V \otimes_k W$ is isomorphic to $\mathcal{M}_{m\times n}(k)$, the space of matrices of size $m \times n$ with entries in k, via an isomorphism that maps $v \otimes w$ to wv, the matrix product of the $m \times 1$ matrix w with the $1 \times n$ matrix v.
   (ii) Use **(i)** to prove there is a k-algebra isomorphism $M_m(k) \otimes_k M_n(k) \to M_{mn}(k)$.

3. Let $A \in M_r(k)$ and let $\lambda_1, \cdots, \lambda_r$ be the eigenvalues of A and $B \in M_s(k)$ with $\mu_1, \cdots, \mu_s$ the eigenvalues of B. Prove that the elements $\lambda_i\mu_j$, $1 \leq i \leq r,\ 1 \leq j \leq s$ are the eigenvalues of $A \otimes B$.

4. Let k be a field and $M_r(k)$ be the matrix algebra. Let $A \in M_m(k)$ and $B \in M_n(k)$ be both nilpotent matrices. Prove that $A \otimes B \in M_{mn}(k)$ is nilpotent, here we identify $M_m(k) \otimes_k M_n(k)$ with $M_{mn}(k)$ via the isomorphism indicated in Problem 2-**(ii)**.

5. Let k be a field. Let $A \in M_m(k)$ and $B \in M_n(k)$ have $det(A) = a$ and $det(B) = b$. Compute $det(A \otimes B)$.