LINEAR ALGEBRA FINAL EXAM

This exam is of **50 marks** and is **3 hours long** - from 10 am to 1pm. Please **read all the questions carefully**. Please feel free to use whatever theorems you have learned in class after stating them clearly.

1. Let A be an $n \times n$ *diagonal* matrix with characteristic polynomial

$$p_A(x) = \prod (x - \lambda_i)^{d_i}$$

where the λ_i are distinct. Let V be the set of matrices B such that

$$AB = BA$$

- Show that V is a vector space (3)
- What is dim(V)? (5)

2. Let V be the vector space $C^0([-1,1], \mathbb{R})$ of continuous real valued functions on $[-1,1]$. Let W_e be the subspace of *even* functions - that is, $f(-x) = f(x)$ and W_o be the subspace of *odd* functions $f(-x) = -f(x)$.

- Show that $V = W_o \oplus W_e$ (4)
- If T is the operator

$$T(f)(x) = \int_0^x f(t)dt$$

then are W_o and W_e invariant under T? Justify your answer. (4)

3. Let $V = \mathcal{M}_n(\mathbb{C})$. Let $A \in V$.
Define a linear operator

$$T_A(B) = AB - BA$$

- Show that if A is a nilpotent matrix then T_A is a nilpotent operator. (5)

Define

$$f_A(B) = Trace(A^t B)$$

- Show that f_A is linear functional on V. (3)
- Show that every linear functional on V is given in this way. (3)

Define a linear operator $M_A : V \to V$ by

$$M_A(B) = ABA^*$$

where $A^* = \bar{A}^t$, the transpose conjugate.

- Show that (5)

$$\det(M_A) = |\det(A)|^{2n}$$

4. Let T be a linear operator on $\mathbb{R}^3$ represented by the matrix

$$A = \begin{pmatrix} 3 & -4 & -4 \\ -1 & 3 & 2 \\ 2 & -4 & -3 \end{pmatrix}$$

- Find vectors $\alpha_1, \ldots, \alpha_r$ such that (5)

$$\mathbb{R}^3 = Z(\alpha_1, T) \oplus \cdots \oplus Z(\alpha_r, T)$$

- Find the rational form of T. (5)
- Compute the minimal polynomial of T (2)
- Compute the characteristic polynomial of T. (2)
- Is T triangulable? (2)
- Is T diagonalisible? (2)