MMath-I, Linear Algebra (Final Test)

Instructions: Total time 3 Hours. Maximum marks 50. **All questions are compulsory**. Refrain from copying from a classmate or any internet source/book. You may use results proved in the class without proof. Use concepts, notations, terminology, results, as covered in the course. If you wish to use a problem as a result, supply its solution too.

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Question 1. Let $(V, \langle , \rangle)$ be a finite dimensional inner product space over k, where $k = \mathbb{R}$ or $\mathbb{C}$. Let V^* denote the dual space of V. Prove that for every $f \in V^*$, there exists a unique vector $w_f \in V$ such that $f(v) = \langle v, w_f \rangle$ for all $v \in V$. (4)

Question 2. Consider $\mathbb{C}$ as an algebra over $\mathbb{R}$. Prove that $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C} \times \mathbb{C}$ as algebras over $\mathbb{C}$. (5)

Question 3. **(i)** Prove that any matrix $A \in GL_n(\mathbb{C})$ satisfying $A^6 = I$ is diagonalizable.

(ii) Determine the number of similarity classes of matrices $A \in GL_3(\mathbb{Z}/2\mathbb{Z})$ that satisfy $A^6 = I$, where I is the identity matrix. (6+10)

Question 4. Let q be power of an odd prime and $\mathbb{F}_q$ be the field having q elements. For a matrix A, let χ_A and m_A respectively denote its characteristic polynomial and minimal polynomial over $\mathbb{F}_q$. Compute the number of elements in the set $\mathcal{C} = \{A \in GL_2(\mathbb{F}_q) | \chi_A = m_A \text{ and } m_A \text{ is reducible}\}$. (15)

Question 5. Let U, V, W be finite dimensional vector spaces over a field k. Assume given a nondegenerate bilinear form $B : V \times V \longrightarrow k$. Prove that there is a natural (i.e. free from any choice of bases) isomorphism of vector spaces $Hom_k(U \otimes V, W) \cong Hom_k(U, V \otimes W)$. (10)