Midterm Exam - Functional Analysis
M. Math I

08 March, 2022

(i) Duration of the exam is 3 hours.

(ii) The maximum number of points you can score in the exam is 100.

(iii) You may directly invoke results proved in the class.

(iv) You are not allowed to consult any notes or external sources for the exam.

Name: ______________________________

Roll Number: ______________________________

1. (10 points) Let X be a measure space. Suppose $f \in L^1(X;\mu)$. Prove that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\int_E |f|\, d\mu < \varepsilon$ whenever $\mu(E) < \delta$.

Total for Question 1: 10

2. Let m denote the Lebesgue measure on $\mathbb{R}$, and $\mathbb{R}_{>0}$ denote the set of positive real numbers.

   (a) (10 points) Prove that, if $1 \leq p < q \leq \infty$, then neither of the spaces $L^p(\mathbb{R};m)$, $L^q(\mathbb{R};m)$ is contained in the other.

   (b) (10 points) Let $0 < \alpha \leq \beta < \infty$. Find all values of p in $[1,\infty]$ in terms of α, β (with justification) such that
   $$\frac{1}{x^\alpha + x^\beta},$$
   belongs to $L^p(\mathbb{R}_{>0};m)$?

Total for Question 2: 20

3. Let $n \in \mathbb{N}$. For a nonempty closed convex set $K \subseteq \mathbb{R}^n$, the *support function* $h(K, \cdot) = h_K : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ is defined by

$$h(K, \vec{u}) := \sup\{\langle \vec{x}, \vec{u} \rangle \mid x \in K\} \text{ for } \vec{u} \in \mathbb{R}^n.$$

   (a) (10 points) Show that the range of h_K is in $\mathbb{R}$ if and only if K is compact.

   (b) (15 points) If $f : \mathbb{R}^n \to \mathbb{R}$ is a sublinear function, then there is a unique non-empty compact convex set K in $\mathbb{R}^n$ such that $f = h_K$.

Total for Question 3: 25

4. Let $C(\mathbb{R})$ denote the space of real-valued continuous functions on $\mathbb{R}$. For a continuous function $\varepsilon : \mathbb{R} \to \mathbb{R}_{>0}$ and $f \in C(\mathbb{R})$, define

$$B(f; \varepsilon) := \{g \in C(\mathbb{R}) : |g(x) - f(x)| \leq \varepsilon(x) \ \forall x \in \mathbb{R}\}.$$

   (a) (10 points) Show that $\{B(f; \varepsilon) : f \in C(\mathbb{R}), \varepsilon \in C(\mathbb{R}; \mathbb{R}_{>0})\}$ is a base for a locally convex topology $\mathcal{T}$ on $C(\mathbb{R})$.

   (b) (10 points) Show that $(C(\mathbb{R}), \mathcal{T})$ is not metrizable.

Total for Question 4: 20

5. (10 points) For finite-dimensional vector space V over $\mathbb{K}$ ($= \mathbb{R}$ or $\mathbb{C}$), show that there is a unique locally convex topology on V.

Total for Question 5: 10

6. (15 points) Prove that every non-empty proper closed convex set in a locally convex space V over $\mathbb{K}$ ($= \mathbb{R}$ or $\mathbb{C}$) is the intersection of some family of half-spaces of the form $\{\vec{x} \in V : \operatorname{Re} \rho(\vec{x}) \leq c\}$, where ρ is a continuous linear functional on V and $c \in \mathbb{R}$.

Total for Question 6: 15