Back Paper | ISI Bangalore | December 2022

Differential Geometry I | 100 Points

Notes.

(a) You may freely use any result proved in class or in the textbook unless you have been asked to prove the same. Use your judgement. All other steps must be justified.

(b) $\mathbb{R}$ = real numbers.

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1. [20 points] Let S be an oriented n-surface in $\mathbb{R}^{n+1}$. Define the Weingarten map at a point p in S. Prove that it is self-adjoint.

2. [20 points] Consider the surface S in $\mathbb{R}^3$ given by the parametrization

$$\sigma(x, y) = (f(x)\cos y,\ f(x)\sin y,\ g(x))$$

where $f(x), g(x)$ are smooth functions in x satisfying $f > 0$ and $(f')^2 + (g')^2 = 1$. Find a unit normal vector field on S. Calculate the first and the second fundamental forms of S in the x, y coordinates and calculate the principal curvatures.

3. [20 points] Let S be a compact oriented n-surface in $\mathbb{R}^{n+1}$. Prove that there exists a point $p \in S$ such that the second fundamental form of S at p is positive definite.

4. [20 points] Let S be a 2-dimensional surface in $\mathbb{R}^3$ and let $\alpha\colon I \to S$ be a geodesic with $\dot{\alpha} \neq 0$. Let $\mathbf{X}$ be a smooth vector field tangent to S along α such that $\|\mathbf{X}\|$ and the angle between $\mathbf{X}$ and $\dot{\alpha}$ are constant along α. Prove that $\mathbf{X}$ is parallel along α.

5. [5+5+6+4=20 points] Consider $\mathbb{R}^3$ with coordinate functions x, y, z. Let $f = xz - y^3$, $g = xy^2 + zy^2$, $h = x + y + z$. Let $\omega = df \wedge dg$.
   (i) Express $\omega \wedge dh$ in terms of the wedge (exterior) products of dx, dy, dz.
   (ii) Compute $d(f dg) - \omega$.
   (iii) Prove that there does not exist a smooth 1-form η such that $d\eta = h\omega$.
   (iv) For the parametrised curve $\beta(t) = (t, \cos t, \sin t)$, compute $\beta^* df$ in terms of dt.