End-Semester Exam — ISI Bangalore — November 2022

Differential Geometry I — 100 Points

Notes.
(a) You may freely use any result proved in class or in the textbook unless you have been asked to prove the same. Use your judgement. All other steps must be justified.
(b) $\mathbb{R}$ = real numbers.

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1. [5+5+5+5=20 points] Let S be a regular surface in $\mathbb{R}^3$. Let $p \in S$ and let $H := p + T_pS$ denote the tangent plane through p. Suppose $C := S \cap H$ is a regular curve with unit-speed parametrisation $\alpha(t)$ such that $\alpha(0) = p$. Prove or disprove each statement below:
   (i) S has normal curvature 0 along $\alpha'(0)$ at p.
   (ii) $\alpha(t)$ is a geodesic on S.
   (iii) The binormal of C at p and the normal to S at p are parallel.
   (iv) C lies in a normal section of S through p.

2. [20 points] Consider the points $p = (0, 0, 1)$, $q = (1, 0, 0)$ and $r = (0, 1, 0)$ on the unit sphere $\mathbb{S}^2$ in $\mathbb{R}^3$. Consider the piecewise smooth path $\alpha(t)$ on $\mathbb{S}^2$ obtained by travelling first from p to q, then to r and then back to p, each of the three segments being the shortest unit-speed geodesic path between the corresponding points. Calculate the parallel transport of the tangent vector $(1, 1, 0)$ at p along α.

3. [20 points] Calculate the Gaussian curvature at an arbitrary point (x, y, z) of the hyperboloid $-x^2 + y^2 + z^2 = 1$.

4. [20 points] Consider the 3-manifold S in $\mathbb{R}^4$ given by the parametrization
$$\sigma(x, y, z) = (f(x)\cos y,\ f(x)\sin y,\ g(x),\ z)$$
where $f(x), g(x)$ are smooth functions in x satisfying $f > 0$ and $(f')^2 + (g')^2 = 1$. Find a unit normal vector field on S. Calculate the first and the second fundamental forms of S in the x, y, z coordinates and calculate the principal curvatures.

5. [5+5+6+4=20 points] Consider $\mathbb{R}^3$ with coordinate functions x, y, z. Let $f = xy + z^2$, $g = xz + yz$, $h = x + y + z$. Let $\omega = df \wedge dg$.
   (i) Express $\omega \wedge dh$ in terms of the wedge (exterior) products of dx, dy, dz.
   (ii) Evaluate $\omega \wedge dh$ on the triple of tangent vectors (v_1, v_2, v_3) at $p = (0, 1, 2)$ where $v_1 = (1, 1, 0)$, $v_2 = (1, 0, 1)$, $v_3 = (0, 1, 1)$.
   (iii) Prove or disprove:
      (a) There exists a smooth 1-form η such that $d\eta = \omega$.
      (b) There exists a smooth 1-form η such that $d\eta = h\omega$.
   (iv) For the parametrised curve $\beta(t) = (\cos t, \sin t, t)$, compute $\beta^* dh$.