Indian Statistical Institute, Bangalore

M. Math First Year

Second Semester - Complex Analysis

Midterm Exam — Date: February 28, 2018

Maximum marks: 100 — Duration: 3 hours

Remark: Each question carries 20 marks. Answer any five.

1. (a) If $\{b_n\}$ is a sequence of non-negative real numbers, then show that $\sup\{x \geq 0 : \lim_{n\to\infty} b_n x^n = 0\} = \frac{1}{\limsup_{n\to\infty} b_n^{\frac{1}{n}}}$.

   (b) Use (a) to prove that the radius of convergence R of a power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ is given by the formula $R = \frac{1}{\limsup_{n\to\infty} |a_n|^{\frac{1}{n}}}$.

   (c) Give an example to show that a power series may or may not converge at a point on the boundary of its disc of convergence.

2. (a) Using the power series definition of the exponential function, show that the function $x \mapsto \exp(ix)$ is a continuous group homomorphism from the additive group $\mathbb{R}$ into the multiplicative group S^1.

   (b) Hence show that the homomorphism of (a) is onto.(Hint: What are the connected subsets of S^1 ?)

3. Let $\Omega = \{z \in \mathbb{C} : Re(z) > 0\}, \Omega' = \{z \in \mathbb{C} : Re(z) > -1\}$.

   (a) Show that the formula $\Gamma(z) = \int_0^\infty e^{-t}\, t^{z-1} dt$ defines a holomorphic function on Ω.

   (b) Show that Γ satisfies $\Gamma(z+1) = z\Gamma(z)\ \forall z \in \Omega$.

   (c) Using (b) or otherwise, show that Γ has an analytic continuation to Ω', except for a pole at 0.

4. (a) Let f be a holomorphic function on a domain Ω. Define $g : \Omega \times \Omega \to \mathbb{C}$ by $g(z,w) = \frac{f(z)-f(w)}{z-w}$ if $z \neq w, g(z,z) = f'(z)$. Show that g is continuous.

   (b) Using (a), show that if $z_0 \in \Omega$ is such that $f'(z_0) \neq 0$, then there is a neighborhood of z_0 on which f is one-one.

   (c) Give an example to show that even if $f'(z) \neq 0$ for all $z \in \Omega, f$ may not be one-one on Ω.

5. (a) If f is a bounded holomorphic function on $\mathbb{C}$ then show that f is a constant function.

   (b) Hence show that $\mathbb{C}$ is algebraically closed.

6. (a) Let $z_0 \in \Omega$. Suppose f is holomorphic on $\Omega \setminus \{z_0\}$. Suppose there is a constant α, $0 \leq \alpha \leq 1$ such that $\lim\limits_{z \to z_0} |z - z_0|^{\alpha}.|f(z)| = 0$. Then show that z_0 is a removable singularity of f.

   (b) Give an example to show that the conclusion of (a) is false for all $\alpha > 1$.

   (c) Give an explicit example of a holomorphic function f with an essential singularity z_0. Prove by direct computation that your example maps each punctured neighborhood of z_0 onto a dense subset of the complex plane.