BACKPAPER EXAMINATION
M. MATH I Year, I SEMESTER
2005-2006
COMPLEX ANALYSIS

Max. Marks:100 DURATION: 3hrs

1. Prove that $\phi(z) = \frac{z-2}{2z+i}$ maps the circle $\{z : \left|z - \frac{1}{2}\right| = \frac{1}{2}\}$ onto a circle (No need to specify the centre and radius; you may use any theorem proved in class) [10]

2. Prove that (the so-called Riemann Zeta function) $\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$ is holomorphic in $\{z : \mathrm{Re}(z) > 1\}$ [10]

3. Say that a region Ω has property EL (existence of lagarithms) if every holomorphic function $f : \Omega \to \mathbb{C}\backslash\{0\}$ has a holomorphic logarithm. Prove that the union of two regions with property EL need not have property EL even if it is a region. What happens if you assume that the intersection of the two regions is connected? [15]

4. If $u(x,y) = e^x \sin(y)$, find a function $v(x,y)$ on $\mathbb{R}^2$ such that $u(x,y) + iv(x,y)$ is an entire function of $x + iy$. [Show how you arrived at v]. [15]

5. Show using contour integration that $\int_{-\infty}^{\infty} \frac{e^{itx}}{1+x^2} dx = \pi e^{-|t|}$ for any real number t. [15]

6. Let $(X, \mathcal{F}, \mu)$ be a finite measure space and Ω an open set in $\mathbb{C}$. Let $\phi : X \to \mathbb{C}$ be a measurable function whose range is disjoint from Ω. If $f(z) = \int_X \frac{1}{\phi - z} d\mu$ show that $f \in H(\Omega)$. [15]

7. Let D be an open disk whose closure $\bar{D}$ is contained in a region Ω. If $f(H(\Omega)), |f|$ is a constant on the boundary of D and f has no zeros in D, show that f is a constant. [hint: use Maximum Modulus Principle] [10]

8. Let Ω be a simply connected region and $f : \Omega \to \mathbb{C}\backslash\{0\}$ be holomorphic. Show that for any positive integer n there exists a holomorphic function g on Ω such that $g^n = f$ on Ω. How many such functions are there? [10]