ANALYSIS OF SEVERAL VARIABLES BACKPAPER EXAMINATION

Total marks: 100
Attempt any SIX questions.
Time: 3 hours

(1) State the Maximum Value Theorem. Show that the rectangular box of maximum volume with a given surface area is a cube. (5+5 = 10 marks)

(2) State the Lagrange Multiplier Theorem. Find the shortest possible distance from the ellipse $x^2 + 2y^2 = 2$ to the line $x + y = 2$. (5+5 = 10 marks)

(3) State the Inverse Function Theorem. Determine at which points $P = (a, b, c)$ the function $f(x, y, z) = (x + y + z, xy + xz + yz, xyz)$ has a local C^1 inverse g, and calculate $Dg(f(P))$. (5+5 = 10 marks)

(4) State the Implicit Function Theorem. Check that the equation $F(x_1, x_2, y) = e^{x_1 y} + y^2 \cos(x_1 x_2) - 1 = 0$ defines y locally as a C^1 function $\phi(x_1, x_2)$ near the point $(1, 2, 0)$, and calculate $D\phi(1, 2)$. (5+5 = 10 marks)

(5) Define the notion of a saddle point of a differentiable function $f : \mathbb{R}^n \to \mathbb{R}$. Find and classify all the critical points of $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = x^3 + y^2 - 6xy$. (5+5 = 10 marks)

(6) State Fubini's Theorem. Is there an integrable function on a rectangle $[a, b] \times [c, d] \in \mathbb{R}^2$ neither of whose iterated integrals exist? (5+5 = 10 marks)

(7) State the Change of Variables Theorem. Let S be the plane region in the first quadrant bounded by the curves $y = x$, $y = 2x$, $xy = 3$ and $xy = 1$. Evaluate $\int_S \frac{x}{y} dA$. (5+5 = 10 marks)

(8) Let C be the curve which is the intersection of the unit sphere and the plane $x + y + z = 0$ in $\mathbb{R}^3$, oriented anticlockwise when viewed from high above the xy-plane. Evaluate the line integral $\int_C y dx$. (10 marks)

(9) Let S be the unit sphere oriented with outward-pointing normal. Calculate $\int_S x dy \wedge dz$ by parametrizing S with spherical coordinates $g(\phi, \theta) = (\sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), \cos(\phi))$ (10 marks)

(10) State Stokes's Theorem. Let S be the sphere $x^2 + y^2 + (z - 1)^2 = 1$ in $\mathbb{R}^3$ oriented with outward unit normal. Evaluate $\int_S \omega$ where $\omega = xz dy \wedge dz + yz dz \wedge dx + z^2 dx \wedge dy$. (5+5 = 10 marks)