ANALYSIS OF SEVERAL VARIABLES FINAL EXAMINATION

Total marks: 50
Attempt all questions.
Time: 3 hours

(1) Let $\Omega \subset \mathbb{R}^3$ be the portion of the unit cube $0 \leq x, y, z \leq 1$ lying between the planes $y + z = 1$ and $x + y + z = 2$. Evaluate $\int_\Omega x dV$. (10 marks)

(2) Let a_n be the volume of the n-dimensional unit ball $B(0; 1) \subset \mathbb{R}^n$. Prove that if $n = 2m$ then $a_n = \frac{\pi^m}{m!}$, and if $n = 2m + 1$ then $a_n = \frac{\pi^m 2^{2m+1} m!}{(2m+1)!}$. (10 marks)

(3) Parametrize the unit sphere S^2 (except for the north pole) by stereographic projection from the north pole as follows. If $(u, v, 0)$ is the point where the line through $(0, 0, 1)$ and (x, y, z) (on the sphere) intersects the plane $z = 0$, solve for u and v. Then solve for $g(u, v) = (x, y, z)$. Let $\omega = x dy \wedge dz$, let S^2 be oriented with the outward pointing normal. Prove that g is an orientation reversing map. Calculate $\int_{S^2} \omega$ using the parametrization given by g. (5+5 = 10 marks)

(4) Let C be the intersection of the cylinder $x^2 + y^2 = 1$ and the plane $2x + 3y - z = 1$ in $\mathbb{R}^3$, oriented clockwise as viewed from high above the xy plane. Evaluate $\int_C y dx - 2z dy + x dz$ directly and by applying Stokes's Theorem. Also find the area of the surface S which is the intersection (in $\mathbb{R}^3$) of the solid cylinder $x^2 + y^2 \leq 1$ and the plane $2x + 3y - z = 1$. (3+3+4 = 10 marks)

(5) Let V be the solid cylinder $x^2 + y^2 \leq 1, -1 \leq z \leq 1$ in $\mathbb{R}^3$. Orient V using the outward normal. Let S be the boundary of V with the induced orientation. Let F be the vector field $F(x, y, z) = (x, y, z)$ on $\mathbb{R}^3$. Calculate the flux $\int_S \mathrm{F.n}\, dS$ of F outward across the surface S both directly and using Stokes's Theorem. Here n is the unit normal vector as determined by the orientation. (5+5 = 10 marks)