MIDTERM: ALGEBRA II

Date: **22th February 2024**
The Total points is **105** and the maximum you can score is **100** points.

(1) (8+8+8+8+8=40) Mark **all correct** options.
  (a) The composition factors of a finite nontrivial solvable group are:
    (i) cyclic groups
    (ii) simple groups
    (iii) Alternating groups
    (iv) cyclic groups of prime orders
  (b) The number of groups of order 35 up to isomorphism are:
    (i) 0
    (ii) 1
    (iii) 2
    (iv) 3
  (c) Which of the following field extensions are algebraic?
    (i) $\mathbb{Q}(\pi/24)/\mathbb{Q}$
    (ii) $\mathbb{Q}(\text{Cos}(\pi/24))/\mathbb{Q}$
    (iii) $\mathbb{C}/\mathbb{Q}$
    (iv) $\mathbb{C}/\mathbb{R}$
  (d) Which of the following field are the splitting field of $X^{11} - 3$?
    (i) $\mathbb{Q}(\sqrt[11]{3})$
    (ii) $\mathbb{Q}(\sqrt[11]{3}, e^{4\pi i/11})$
    (iii) $\mathbb{Q}(\sqrt[11]{3}, \text{Cos}(2\pi/11))$
    (iv) $\mathbb{Q}(\sqrt[11]{3}, \text{Cos}(2\pi/11), i\,\text{Sin}(2\pi/11)$
  (e) Which of the following extensions are splitting fields over the base field?
    (i) $\mathbb{F}_{125}/\mathbb{F}_5$ where $\mathbb{F}_{125}$ is a degree 3 extension of $\mathbb{F}_5$.
    (ii) $\mathbb{Q}(\sqrt[3]{7})/\mathbb{Q}$
    (iii) $\mathbb{R}/\mathbb{Q}$
    (iv) $\mathbb{Q}(\sqrt[4]{3})/\mathbb{Q}(\sqrt{3})$

(2) (5+15=20 points) What is meant by a group G acts on a set S. Let p be a prime number and G be a group of order a multiple of p. If G contains a subgroup of index strictly less than p then show that G is not simple.

(3) (20 points) Let G be a finite group of order pqr where p, q and r are distinct primes. Show that G is solvable.

(4) (10 points) Let K/F be an algebraic extension. Let R be a subring of K containing F. Show that R is a field.

(5) (15 points) Let $a = \sqrt{7} + \sqrt[3]{5}$. Compute the minimal polynomial $p(x)$ of a over $\mathbb{Q}$. Let K be the splitting field of $p(x)$ over $\mathbb{Q}$. Compute $[K : \mathbb{Q}]$.