Back Paper

Time:	10:00–13:00, June 9, 2022.	Course name:	*Algebra II*
Degree:	MMath.	Year:	1st Year, 2nd Semester; 2021–2022.
Course instructor:	Ramdin Mawia.	Total Marks:	60.

Attempt four of the following problems, including problems nº 3 & nº 4.

Group Theory

1. Describe all groups of order 20 (up to isomorphism). **15**

2. Define a solvable group. Show that any group of order p^2q is solvable, where $p < q$ are odd primes. **15**

3. Decide whether the following statements are true or false, with brief but complete justifications (**any five**): **15**

   (a) If G is a cyclic group of order n and d is a divisor of n, then G has a subgroup of order d.

   (b) Every group of order 51 is cyclic.

   (c) If a finite group G acts transitively on a finite set X, then $|X|$ divides $|G|$.

   (d) The number of Sylow p-subgroups of $\mathrm{GL}_2(\mathbb{F}_5)$ is 6.

   (e) If $1 \to F \to E \to G \to 1$ is a short exact sequence of groups with F and G cyclic of prime order, then E cannot be cyclic.

   (f) If P and Q are Sylow p-subgroups of G with $|P| = |Q| = p^2$ and $|P \cap Q| \geqslant p$, then $P = Q$.

Galois Theory

4. When do we say that a field extension is separable? **15**

   (a) Define the separable degree and prove that it is bounded by the degree of the extension (for finite extensions).

   (b) Is it true that every finite extension of a finite field is separable? Justify.

5. State and prove the Fundamental Theorem of Galois Theory. **15**

6. Find the Galois groups of two of the following polynomials, with justifications: **15**

   (a) $X^4 + 3X + 6 \in \mathbb{Q}[X]$.

   (b) $X(X^2 + 134)(X^2 - 16) + 2 \in \mathbb{Q}[X]$.

   (c) $X^3 + X + 3 \in \mathbb{Q}[X]$.

7. State whether the following statements are true or false, with brief but complete justifications (**any five**): **15**

   (a) The multiplicative group of a finite field is cyclic.

   (b) If K/k is a cyclic extension of degree n, then K contains a primitive nth root of unity.

   (c) The angle $\pi/13$ is constructible.

   (d) Roots of the polynomial $X(X^2 + 14)(X^2 - 4) + 2 \in \mathbb{Q}[X]$ are expressible in radicals.

   (e) Let k be a field of characteristic $p > 0$. If some extension of k contains a primitive nth root of unity, then p does not divide n.

   (f) The separable closure of $\mathbb{Q}$ in $\mathbb{C}$ is the same as its algebraic closure.

   (g) -1 is a square in the field $\mathbb{Q}[\sqrt[4]{-3}]$.