Mid-Semester Exam | ISI Bangalore | September 2023

MMath Algebra I | 100 Points

Notes.

(a) You may freely use any result proved in class unless you have been asked to prove the same. Use your judgement. All other steps must be justified.

(b) We use $\mathbb{N}$ = natural numbers, $\mathbb{Z}$ = integers, $\mathbb{Q}$ = rational numbers, $\mathbb{R}$ = real numbers.

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1. [15 points] If a ring R has a nonzero proper ideal, prove that it has a proper ideal which is not a prime ideal.

2. [20 points] Let a, b be nonzero coprime integers. Prove that $\mathbb{Z}[i]/(a+bi)$ is isomorphic to the ring $\mathbb{Z}/(a^2+b^2)\mathbb{Z}$.

3. [20 points] Let $\mathfrak{m}_1$ and $\mathfrak{m}_2$ be distinct maximal ideals in a ring. Prove that for any positive integers s_1, s_2, we have $\mathfrak{m}_1^{s_1} \cap \mathfrak{m}_2^{s_2} = \mathfrak{m}_1^{s_1}\mathfrak{m}_2^{s_2}$.

4. [15 points] Let F be a field and R an F-algebra having vector space dimension 2 over F. Prove that either R is a field or R is isomorphic to $F \times F$ or is isomorphic to $F[x]/(x^2)$.

5. [15 points] Let $f\colon A \to B$ be a map of $\mathbb{C}$-algebras. Prove that for any maximal ideal $\mathfrak{m}$ in B, the inverse image $f^{-1}\mathfrak{m}$ is a maximal ideal in A. (Hint: Is $B/\mathfrak{m} \xrightarrow{\sim} \mathbb{C}$?)

6. [15 points] For every integer $n > 0$, find the number of distinct ways in which 15^n can be written as a sum of two squares.