End-Semester Exam — ISI Bangalore — November 2023

MMath Algebra I — 100 Points

Notes.

(a) You may freely use any result proved in class unless you have been asked to prove the same. Use your judgement. All other steps must be justified.

(b) We use $\mathbb{N}$ = natural numbers, $\mathbb{Z}$ = integers, $\mathbb{Q}$ = rational numbers, $\mathbb{R}$ = real numbers, $\mathbb{C}$ = complex numbers.

(c) There are a total of **105** points in this paper. You will be awarded a maximum of **100** points.

---

1. [$5 \times 8 = 40$ Points] Do any 5 among the following 7 choices (a)–(f).
   In each case, give example(s) as per the required condition.

   (a) A UFD which is not a PID.

   (b) An irreducible element in a domain R which is not a prime element.

   (c) An odd prime number $p \in \mathbb{Z}$ that remains irreducible in $\mathbb{Z}[\omega]$ where $\omega = e^{2\pi i/3}$.

   (d) An idempotent element $e \neq 0, 1$ in the ring $\mathbb{R}[x, y]/(x^2 + 1, y^2 + 1)$.

   (e) Infinitely many ideals I_λ in the ring $\mathbb{Q}[x, y]$ such that $(x, y)^2 \subsetneq I_\lambda \subsetneq (x, y)$.

   (f) A domain R and a torsion R-module M such the annihilator of M is (0).

   (g) A ring R and two nonzero modules M, N such that $\mathrm{Hom}_R(M, N) \cong (0)$.

2. [15 Points] Let C denote the ring of continuous real-valued functions on the real line $\mathbb{R}$.

   (i) Give an example of a maximal ideal in C.
   (ii) Give an example of a non-finitely generated ideal in C.
   (iii) Give an example of a finitely generated C-module M together with finitely many generators, for which the module of relations is not finitely generated.

3. [15 Points] Express as a direct sum of cyclic groups, the cokernel M of the map $\mathbb{Z}^4 \to \mathbb{Z}^4$ given by the following matrix.

$$\begin{pmatrix} 3 & 3 & 3 & 3 \\ 1 & 3 & 1 & 1 \\ 1 & 1 & 4 & 1 \\ 2 & 2 & 2 & 2 \end{pmatrix}$$

4. [15 Points] Let $0 \longrightarrow N \xrightarrow{\ i\ } M \xrightarrow{\ f\ } P \longrightarrow 0$ be an exact sequence of R-modules over a ring R. Prove that if P is free, then there exists a map $\pi\colon M \to N$ such that πi is the identity map on N.

5. [20 Points] Let R be a ring and let M, N be two R-modules.

(i) Define the tensor product of M and N over R in terms of its universal property.

(ii) Give a construction to prove that the tensor product exists. (You must also check that it satisfies the universal property).

(iii) Prove that for any R-module M, there is an isomorphism $R \otimes_R M \cong M$.