Final Exam - Advanced Linear Algebra
M. Math II

24 November, 2022

(i) Duration of the exam is 3 hours.

(ii) The maximum number of points you can score in the exam is 100. (Any score above 100 will be rounded down to 100.)

(iii) You are not allowed to consult any notes or external sources for the exam.

Name: ____________________

Roll Number: ____________________

1. Let A, B be positive operators on a finite-dimensional Hilbert space $\mathscr{H}$ with $A \geq B$ (that is, $A - B$ is positive). For $1 \leq k \leq \dim(\mathscr{H})$, show that:

   (a) (10 points) $\otimes^k A \geq \otimes^k B$;

   (b) (10 points) $\wedge^k A \geq \wedge^k B$;

   (c) (5 points) $\det(A) \geq \det(B)$.

Total for Question 1: 25

2. Let $\mathscr{H}, \mathscr{K}$ be finite-dimensional complex Hilbert spaces, and let $L(\mathscr{H})$ ($L(\mathscr{K})$, respectively) denote the set of linear operators on $\mathscr{H}$ ($\mathscr{K}$, respectively). Note that $L(\mathscr{H}), L(\mathscr{K})$ may be considered as Hilbert spaces equipped with the Hilbert-Schmidt inner product. Let Φ be a linear map from $L(\mathscr{H})$ to $L(\mathscr{K})$, and Φ^* denote the adjoint map.

   (a) (10 points) Show that Φ is positive if and only if Φ^* is positive.

(b) (5 points) Show that Φ is trace-preserving if and only if Φ^* is unital.

(c) (15 points) Show that there are completely positive maps $\Phi_1, \Phi_2, \Phi_3, \Phi_4 : L(\mathscr{H}) \to L(\mathscr{K})$ such that

$$\Phi = (\Phi_1 - \Phi_2) + i(\Phi_3 - \Phi_4).$$

Total for Question 2: 30

3. (20 points) Let A and B be quantum systems with corresponding (finite-dimensional) state spaces $\mathscr{H}_A$ and $\mathscr{H}_B$. Suppose $|\psi\rangle$ and $|\varphi\rangle$ are two pure states of a composite quantum system with components A and B (and state space $\mathscr{H}_A \otimes \mathscr{H}_B$), with identical Schmidt coefficients. Show that there are unitary transformations U on system A ($U \in L(\mathscr{H}_A)$) and V on system B ($V \in L(\mathscr{H}_B)$) such that $|\psi\rangle = (U \otimes V)|\varphi\rangle$.

Total for Question 3: 20

4. Let $\mathbb{C}^n$ denote the standard n-qubit quantum system. For $\lambda \in [0, 1]$, we define the map $D_\lambda : L(\mathbb{C}^n) \to L(\mathbb{C}^n)$ by

$$D_\lambda(X) = (1 - \lambda)\frac{1}{n}Tr(X)I_{\mathbb{C}^n} + \lambda X,$$

for $X \in L(\mathbb{C}^n)$.

(a) (10 points) Compute the Choi representation of D_λ.

(b) (5 points) Show that D_λ is a quantum channel, that is, a trace-preserving, completely positive map. (It is called the *depolarizing channel*.)

Total for Question 4: 15

5. Let $A \in M_n(\mathbb{C})$ and $\Phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be the map $\Phi(X) = A \circ X$ (where $\circ$ denotes the Hadamard product of matrices.)

(a) (10 points) Show that Φ is completely positive if and only if A is a positive-semidefinite matrix.

(b) (10 points) Show that Φ is unital if and only if it is trace-preserving.

Total for Question 5: 20