Nurture Programme 2007-2010 at I.S.I. Bangalore
Problems in number theory - 2nd set

Q 1.
For odd primes $p \neq q$, prove that the quadratic reciprocity law is equivalent to the statement

$$(\frac{p}{q}) = 1 \Leftrightarrow q \equiv \pm\alpha^2 \ mod \ 4p$$

for some odd α.

Q 2.
Show that $(1!)^2 + (2!)^2 + \cdots + (n!)^2$ is not a square when $n > 1$.

Q 3.
Prove that $2^n - 1$ cannot divide $3^n - 1$ for $n > 1$.

Q 4.
Prove that no square can be of the form $x^3 + 7$.

Q 5.
Show that the polynomial $(x^2 - 13)(x^2 - 17)(x^2 - 221)$ has no solutions in integers whereas it has solutions modulo p for every prime p.

Q 6.
Prove that the binary quadratic forms $3x^2 + xy + 4y^2$ and $3x^2 - xy + 4y^2$ are not equivalent forms although they represent the same set of integers.

Q 7.
Let $n \equiv 1$ mod 4 be a nonsquare positive integer such that $n = a^2 + 4b^2$ for some integers a, b. Assume that $x^2 - ny^2 = -1$ has an integer solution $(x, y) = (t_1, t_2)$. Show that $n = rs$ for some r, s where $rx^2 - sy^2 = a$ has an integer solution. What do you deduce when a prime $p > 2$ is expressed as a sum of two squares.

. P.T.O.

Q 8.
Consider the following 2-dimensional analogue $p_2(n)$ of the partition function. Write each rectangular array summing to n where the numbers are non-increasing both from left to right and from top to bottom. For example, $p(4) = 5$ while $p_2(4) = 13$ as the thirteen 2-dimensional partitions of 4 are (apart from $4, 3+1, 2+2, 2+1+1, 1+1+1+1$) :

$$\left|\begin{array}{r} 3 \\ +1 \end{array}\right| \left|\begin{array}{r} 2 \\ +1 \\ +1 \end{array}\right| \left|\begin{array}{r} 1 \\ +1 \\ +1 \\ +1 \end{array}\right| \left|\begin{array}{r} 2 \\ +2 \end{array}\right| \left|\begin{array}{rr} 2 & +1 \\ +1 & \end{array}\right| \left|\begin{array}{rr} 1 & +1 \\ +1 & +1 \end{array}\right| \left|\begin{array}{rrr} 1 & +1 & +1 \\ +1 & & \end{array}\right| \left|\begin{array}{rr} 1 & +1 \\ +1 & \\ +1 & \end{array}\right|$$

We have seen that $1 + \sum_{n \geq 1} p(n)t^n = \prod_{k \geq 1} \frac{1}{1-t^k}$. Find an analogous expression for the generating function $1 + \sum_{n \geq 1} p_2(n)t^n$ of $p_2(n)$. What guess would you make for the generating function higher-dimensional partitions?

Q 9.
Assume (learn this proof, if possible) that for a given natural number n, the number $r_2(n)$ of ordered pairs of integers (x, y) with $x^2 + y^2 = n$ equals $4\#\{d|n, d \equiv 1 \bmod 4\} - 4\#\{d|n, d \equiv 3 \bmod 4\}$. Using this, prove :

$$\sum_{r \geq 0} [\sqrt{n - r^2}] = \sum_{k \geq 0} (-1)^i \left[\frac{n}{2k+1}\right].$$

Q 10.
(i) Show that the above equality for $r_2(n)$ can be rephrased as

$$\sum_{n \geq 1} \frac{r_2(n)}{n^s} = 4\Big(\sum_{n \geq 1} \frac{1}{n^s}\Big)\Big(\sum_{n \geq 1} \frac{(-1)^{n-1}}{(2n-1)^s}.$$

(ii) If $\sum_{n \geq 1} \frac{a_n}{n^s} = (\sum_{n \geq 1} \frac{1}{n^s})(\sum_{n \geq 1} \frac{b_n}{n^s})$, prove that $\sum_{n \geq 1} a_n t^n = \sum_{n \geq 1} \frac{b_n x^n}{1-x^n}$.
(iii) Use (i),(ii) to deduce that the number $r_8(n)$ of ways of expressing n as a sum of 8 squares, equals $16 \sum_{d|n} (-1)^{n+d} d^3$.