Nurture Programme 2007-2010 in I.S.I. Bangalore
Problems in combinatorics - 2nd set

As some problems from the first set were not done by most people, a few problems below cover these topics also.

Q 1.
Recall that a Latin square is a square array in which each row and each column consists of the same set of entries without repetition. Two $n \times n$ Latin squares $L = (l_{ij}), M = (m_{ij})$ are said to be orthogonal, if the n^2 pairs (l_{ij}, m_{ij}) are all distinct. For example, the only two 2×2 Latin squares are $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ and $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$; these are not orthogonal.
On the other hand, define a finite projective plane of order n to be a finite set of 'points' and 'lines' (a collection of subsets of the point-set) satisfying the properties :
each pair of points lies on exactly one line, each pair of lines have exactly one point in common, each line contains $n + 1$ points and each point lies on exactly $n + 1$ lines.
Prove that a projective plane of order n exists if and only if there are $n - 1$ mutually orthogonal $n \times n$ Latin squares.

Q 2.
Prove that a projective plane of order must contains exactly n^2+n+1 points and exactly $n^2 + n + 1$ lines.

Q 3.
Assume the following important result of Euler (learn it if possible) :
A connected graph has an Eulerian path (a closed path using all the edges of the graph) if and only if each vertex has even degree (even number of edges emanating from it). Deduce from this that a connected graph admits an Eulerian tour (a non-closed path using all the edges) if and only if there are precisely two vertices of odd degree.

Q 4.
Show that a knight's route on an $n \times n$ chess board using every possible move just once (in one direction or the other) exists if and only if $n \leq 3$.

Q 5.
Consider a rectangular matrix with 0's and 1's as entries. Show that each set of r rows has 1's in at least r columns if and only if there is a 1 in each row which are in different columns.

Q 6.
Given a subgroup G of S_n, its cycle index is defined as the following polynomial in n variables :

$$Z(G, t_1, \cdots, t_n) = \frac{1}{O(G)} \sum_{g \in G} t_1^{l_1(g)} t_1^{l_2(g)} \cdots t_n^{l_n(g)}.$$

Here $l_i(g)$ denotes the number of i-cycles in g. Compute the cycle index of the following subgroups of S_n : (i) a cyclic subgroup of order n, (ii) the dihedral group D_n of order $2n$ which is the group of rotations of a regular n-gon and, (iii) the alternating group A_n.

Q 7.
Use the R-S-K correspondence to compute the number of $m \times n$ matrices whose entries are integers ≥ 0 with sum equal to r.

Q 8.
Compute the number of $m \times n$ matrices with non-negative integral entries and row sums $r_1, \cdots, r_m$ and column sums $c_1, \cdots, c_n$.

Q 9.
Let A be an $m \times n$ matrix with non-negative integral entries, and consider the 'derived' matrix $A^\sharp$ obtained from A by the matrix-ball construction. If a symmetric matrix A corresponds to a tableau P, show that tr (A)+ tr $(A^\sharp)$ equals the length of the first row of P.