Nurture Programme 2008
Some problems in combinatorics

Some of these problems may be hard. Don't despair if you cannot solve them; such problems will be discussed during the contact programme.

Q 1.
Let $\binom{n}{r}$ stand for the binomial coefficient "n choose r". Prove that

$$d! = \sum_{r=0}^{d} (-1)^r \binom{d}{r} (x+d-r)^d \quad \forall \;\; x.$$

Deduce Wilson's theorem from Fermat's little theorem using this identity.

Q 2.
For each natural number r, denote by S_r, the set of r-digit numbers with the digits in strictly decreasing order. Write the elements of S_r in increasing order. For instance, S_3 consists of $210, 310, 320, 321, 410, 420, 421, 430, 431, 432$ and so on. If a number in S_r has the digits $a_r, a_{r-1}, \cdots, a_1$ prove that the number of numbers which occur in S_r before this number equals

$$\binom{a_r}{r} + \binom{a_{r-1}}{r-1} + \cdots + \binom{a_1}{1}.$$

Hence, argue that for natural numbers n, r, there is a unique expansion

$$n = \binom{a_r}{r} + \binom{a_{r-1}}{r-1} + \cdots + \binom{a_1}{1}$$

with $a_r > a_{r-1} > \cdots > a_1 \geq 0$.

Q 3.
(a) Let $p(n)$ denote the number of partitions of n and, for any $l \leq n$, let $p(n,r)$ denote the number of partitions of n into exactly r parts. For instance the $p(4) = 5$ partitions of 4 are $4, 3+1, 2+2, 2+1+1, 1+1+1+1$ and $p(4,1) = 1, p(4,2) = 2, p(4,3) = 1, p(4,4) = 1$. Prove :

$$\sum_{n=1}^{\infty} p(n) T^n = T \prod_{n=1}^{\infty} (1-T^n)^{-1};$$

$$\sum_{n=1}^{\infty} p(n,r) T^n = \frac{T^r}{(1-T)(1-T^2)\cdots(1-T^r)}.$$

(b) There are 3 partitions $5 = 3+1+1 = 1+1+1+1+1$ of 5 into parts all of which are odd and 3 partitions $5 = 5 = 4+1 = 3+2$ into parts which are all distinct. In general, establish a one-to-one correspondence between the sets of partitions of an integer n into odd parts and of partitions of n into distinct parts.

Q 4.
The Chebychev polynomials of the first kind are defined by $Cos(n\theta) = T_n(Cos\theta)$. The Chebychev polynomials of the second kind are defined by $U_n(Cos\theta) = \frac{Sin(n+1)\theta}{Sin\theta}$. Show :

$$\frac{1-tx}{1-2tx+t^2} = \sum_{n\geq 0} T_n(x)t^n;$$

$$\frac{1}{1-2tx+t^2} = \sum_{n\geq 0} U_n(x)t^n.$$

Q 5.
Consider the transformation $T(n) = n+1$ or $T(n) = \frac{n}{2}$ according as to whether n is odd or n is even. Let us write $T^2(n) = T(T(n)), T^3(n) = T(T(T(n)))$ etc. Prove that for each n, there exists k for which $T^k(n) = 1$. Determine the number F_k of natural numbers n such that that $T^k(n) = 1$.

Q 6.
For natural numbers $n_1, n_2, \cdots, n_r$, prove :

$$max(n_1, \cdots, n_r) =$$

$$\sum_i n_i - \sum_{i<j} min(n_i, n_j) + \sum_{i<j<k} min(n_i, n_j, n_k) - \cdots + (-1)^{r-1} min(n_1, \cdots, n_r).$$

Deduce from this for any sequence of natural numbers $a_1, \cdots, a_r$ that one has :

$$[a_1, \cdots, a_r] = \frac{(\prod_i a_i)(\prod_{i<j<k}(a_i, a_j, a_k))\cdots}{(\prod_{i<j}(a_i, a_j))(\prod_{i<j<k<l}(a_i, a_j, a_k, a_l))\cdots}$$

Q 7.
If every lattice point of the co-ordinate plane is coloured either black or white, prove that there must be some rectangle all of whose sides are parallel to the axes and whose vertices have the same colours.

Q 8.
A Latin square is a square array in which each row and each column consists of the same set of entries without repetition. Two $n \times n$ Latin squares $L = (l_{ij}), M = (m_{ij})$ are said to be orthogonal, if the n^2 pairs (l_{ij}, m_{ij}) are all distinct. For example, the only two 2×2 Latin squares are $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ and $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$; these are not orthogonal. Prove, for every $n \geq 2$ that there are at the most $n - 1$ pairwise orthogonal $n \times n$ Lain squares.

Q 9.
Consider a 3×3 chess board. Show that there is a knight's route on the board which uses every possible move just once (in one direction or the other). For $n \geq 4$, on an $n \times n$ chess board, does the graph of knight moves have an Eulerian path (a closed path which uses all edges of the graph) ?

Q 10.
Let $X = \sqcup_{i=1}^n X_i = \sqcup_{i=1}^n Y_i$ be two disjoint decompositions with all sets X_i's and Y_j's having the same size. Prove that there exist distinct elements $x_1, \cdots, x_n$ which are in different sets in both decompositions.

Q 11.
Let d_n denote the number of ways of obtaining a total of n in successively throwing a die (for example, $d_4 = 8$). Show that d_n equals the coefficient of x^n in $(1 - x - x^2 - x^3 - x^4 - x^5 - x^6)^{-1}$.

Q 12.
Given a $n \times n$ chess-board, consider any subset S of squares on it. Define the rook polynomial of S to be the polynomial

$$r_S(x) = 1 + r_1(S)x + \cdots + r_n(S)x^n$$

where $r_n(S)$ is the number of ways of placing n rooks on squares in S such that no two are threatening (that is, no two are in the same row or column). For example, the rook polynomial of the set S in the following figure (where the squares in S are marked with an S) is $1 + 8x + 19x^2 + 14x^3 + 2x^4$:

S			S
	S	S	S
		S	
S	S		

Let S be any subset of an $n \times n$ chess board and $\overline{S}$, its complement. Prove :

$$r_n(\overline{S}) = n! - (n-1)!r_1(S) + (n-2)!r_2(S) - \cdots + (-1)^n r_n(S).$$

Q 13.

For a prime number p, look at the following $(p-1) \times (p-1)$ matrix with entries in the set of integers modulo p :

$$\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 3 & \cdots & p-1 \\ 1^2 & 2^2 & 3^2 & \cdots & (p-1)^2 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 1^{p-2} & 2^{p-2} & 3^{p-2} & \cdots & (p-1)^{p-2} \end{pmatrix}$$

Find the inverse of this matrix - here, matrix multiplication involves addition and multiplication modulo p and any a^r stands for its value modulo p.