Nurture Programme 2008: Problems in Analysis

1. Show that a set A is infinite if and only if for every finite set B there is a one-one map $\phi\colon B \to A$.

2. Show that $Q[i] = \{\alpha + i\beta \mid \alpha, \beta \text{ are rationals }\}$ is a countably infinite set.

3. Let $\{s_n\}$ be a sequence defined by

$$s_1 = \sqrt{2}, \qquad s_n = \sqrt{2 + \sqrt{s_{n-1}}}, \qquad n \geq 2.$$

   Then show that $\{s_n\}$ converges.

4. Let $\{a_n\}$ and $\{b_n\}$ be two bounded sequences of real numbers. Suppose $a_n \to a$. Then find the realtionship between $\limsup(a_n + b_n)$ and $\liminf(a_n + b_n)$.

5. If $\sum_{n=1}^{\infty} a_n < \infty$ and $a_n \geq 0$, then $\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n} < \infty$.

6. If $\sum_{n=1}^{\infty} a_n < \infty$ and $a_n \geq 0$, then $\sum_{n=1}^{\infty} a_n^2 < \infty$.

7. Assume that $\sum_{r=1}^{n} \frac{1}{(r+2)(r+3)} = \frac{1}{3} - \frac{1}{n+3}$. Find $\sum_{r=1}^{\infty} \frac{1}{(r+2)(r+3)}$.

8. If x is not an integer, prove that $\frac{1}{x+1} - \frac{1}{x+2} + \frac{1}{x+3} - \cdots$ converges. Is this series absolutely convergent.

9. Let x be a real number with $|x| < 1$ and q be another real number. Then show that the series $\sum_{n=0}^{\infty} n^q x^n$ is absolutely convergent and in particular, $\lim n^q x^n = 0$.

10. Determine the possible points of continuity of

$$\begin{aligned} f(x) &= x \quad \text{if } x \text{ is rational} \\ &= 0 \quad \text{otherwise} \end{aligned}$$

11. Find the points of continuity of $f(x) = x - [x]$ for any real x where $[x]$ is the largest integer smaller than or equal to x.

12. Suppose f is a real-valued continuous function on $(0, \infty)$. If $g(x) = f(\frac{1}{x})$, then show that $\lim_{x\to\infty} f(x) = L$ if and only if $\lim_{x\to 0^+} g(x) = L$.

13. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} \frac{1}{2n} & \text{if } \ x = \frac{1}{n}, \quad n \in N \\ \frac{1}{x} & \text{if } \ x = 3, 5, 7, \cdots \\ \frac{x}{2} & \text{if } \ x = 2, 4, 6, \cdots \\ x & \text{otherwise} \end{cases}$$

Show that f is bijective and f is continuous at 0 but f^{-1} is not continuous at $f^{-1}(0)$.

14. Let $a, b \in \mathbb{R}$ with $a < b$. Let $f\colon [a,\ b] \to \mathbb{R}$ be a increasing function. Suppose f maps $[a,\ b]$ onto $[f(a),\ f(b)]$. Then show that f is continuous. Give a counter-example to show that this need not hold if f is not increasing.

15. Show that a continuous rational valued function must be constant.

16. If $f\colon [0,\ \infty) \to \mathbb{R}$ is a continuous function such that $\lim_{x\to\infty} f(x) = 10$, then show that f is uniformly continuous.

17. Show that $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = d(x,\ \mathbb{N}) = \inf_{n\geq 1}|x - n|$ is uniformly continuous.

18. Let f be a real-valued continuous function on a bounded interval $[a,\ b]$. Suppose $s, t \in f([a,\ b])$ and $s \neq t$. Then prove that $\delta(s,\ t) = \inf\{|x - y| \mid f(x) = s, \quad f(y) = t\}$ is positive. Further if $r, s, t \in f([a,\ b])$ with $r < s < t$, then show that $\delta(r, s) < \delta(r, t)$. That is, fixing r, show that the map $s \mapsto \delta(r, s)$ is an increasing function in $f([a,\ b]) \cap (r, \infty)$.

19. Let $a, b \in R$ with $a < b$. Suppose $f\colon [a,\ b] \to \mathbb{R}$ is a continuous one-one function. Then show that f is strictly monotone.

20. Suppose $\{x_n\}$ is a Cauchy sequence in $[0,\ 1)$ and has convergent subsequence in $[0,\ 1)$. Then show that $x_n \to x \in [0,\ 1)$.