Assignment in Algebra

1. Let p be a prime integer. Determine the number of p-Sylow groups in the permutation group on p letters.

2. Show that every subgroup of order < 60 is solvable.

3. Show that a finite abelian group is not cyclic if and only if it contains $\mathbb{Z}_p \times \mathbb{Z}_p$ as a subgroup, for some prime integer p.

4. Let S be a set with at least two elements and G be a group acting transitively on S. For $x \in G$, let $f(x)$ denote the number of elements of S that are fixed by x. Prove that

$$\sum_{x \in G} f(x) = |G|$$

where $|G|$ denote the cardinality of G.

5. Show that no group of order 224 is simple.

6. Prove that any prime ideal in the polynomial ring $\mathbb{Z}[X]$ can be generated by two generators. What can you say about the number of generators of a non-prime ideal in $\mathbb{Z}[X]$?

7. Let $A = [x_{ij}]$ be the $n \times n$ matrix of variables. Prove that the determinant of A is an irreducible polynomial in the polynomial ring $\mathbb{Z}[x_{ij}]$.

8. Prove that the quotient ring $\mathbb{C}[X, Y]/(X^2 + Y^2 - 1)$ is a unique factorization domain but the ring $\mathbb{R}[X, Y]/(X^2 + Y^2 - 1)$ is not a unique factorization domain.

9. Identify the quotient field of the ring $\mathbb{C}[X, Y]/(X^2 + Y^2 - 1)$.

10. Let R be a quotient ring of the polynomial ring $\mathbb{C}[X]$. Prove that for any maximal ideal M of R, the ring R/M is isomorphic with $\mathbb{C}$.