Nurture programme 2007-2011
Algebra assignment III
September 2009

Q 1.
(a) Let A be a PID and K, its quotient field. If $A \subset B \subset K$ with B a subring, prove that B is also a PID.
(b) Prove that every ideal in $\mathbf{Z}[X]$ is finitely generated.
(c) Prove that any prime ideal in $\mathbf{Z}[X]$ can be generated by at most two elements.
(d)* Prove that every ideal in $\mathbf{Z}[X]$ can be generated by at most two elements.

Q 2.
(a) Prove that any complex $n \times n$ matrix A and its transpose are similar; that is, there is an invertible matrix P with $PAP^{-1} = A^t$.
(b)* Prove that any $n \times n$ matrix A over rational numbers is similar over $\mathbf{Q}$ to A^t; that is, there is an invertible matrix P with entries in $\mathbf{Q}$ such that $PAP^{-1} = A^t$.
(c) Given a matrix $A \in M_n(\mathbf{Q})$, show that there is an invertible matrix P with entries in $\mathbf{C}$ such that PAP^{-1} is upper triangular.

Q 3.

(a) If P is an $n \times n$ integer matrix with det $P = \pm 1$, show that the mapping $\theta_P : \mathbf{Z}^n \to \mathbf{Z}^n$ given by $\theta_P(v_1, \cdots, v_n) = P \begin{pmatrix} v_1 \\ v_2 \\ : \\ : \\ v_n \end{pmatrix}$ is an isomorphism of groups.

Hint : P has an inverse matrix which has integer entries as well.

(b)* Let A be an $m \times n$ matrix with integer entries with $m \leq n$. Show that there is an $m \times m$ integer matrix P and det $P = \pm 1$ and an $n \times n$ integer matrix Q and det $Q = \pm 1$ so that PAQ has the form of a diagonal $m \times m$ matrix $diag(d_1, \cdots, d_m)$ with integers $d_i | d_{i+1}$ and the last $n - m$ columns to be zeroes.

(c) For a matrix $A \in M_n(\mathbf{Z})$ with det $A \neq 0$, consider the group homomorphism $\theta_A : \mathbf{Z}^n \to \mathbf{Z}^n$ given by $\theta_A(v_1, \cdots, v_n) = A \begin{pmatrix} v_1 \\ v_2 \\ : \\ : \\ v_n \end{pmatrix}$ as before.

Using (a) and (b) with $m = n$, show that the image of θ_A, has finite index equal to $|detA|$.

(d) Use this to compute the cardinality of the quotient ring $\mathbf{Z}[i]/(a + bi)$ for any $a + bi \in \mathbf{Z}[i]$.

Q 4.

Let A be a commutative ring and let

$$0 \to M_1 \to M_0 \to M_2 \to 0$$

be an exact sequence of A-modules - this means that the A-module homomorphism $\alpha : M_1 \to M_0$ is injective, $\beta : M_0 \to M_2$ is surjective and $Ker(\beta) = Image(\alpha)$.

Prove that there is an A-module homomorphism $s : M_2 \to M_0$ with $\beta \circ s = Id_{M_2}$ if, and only if, $M_0 \cong M_1 \oplus M_2$.

Q 5.

(a) If V is an n-dimensional vector space and T is a nilpotent endomorphishm of V, then show that the n-th power of T is zero.

(b) Use the Chinese remainder theorem for the polynomial ring $\mathbf{C}[X]$ to show the following. For any $A \in M_n(\mathbf{C})$, there is a polynomial $f \in X\mathbf{C}[X]$ such that $f(A)$ is nilpotent and $A - f(A)$ is diagonalizable (i.e. conjugate to a diagonal matrix).

Q 6.

Let V be a finite-dimensional vector space over any field K and let $T \in EndV$. Consider V as a $K[X]$-module by the action

$$(\sum_{i=0}^{n} a_i X^i)(v) := \sum_{i=0}^{n} a_i T^i(v).$$

Using this, or otherwise, prove that there exists a basis of V with respect to which T has a matrix of the form

$$\begin{pmatrix} c(f_1) & & & \\ & c(f_2) & & \\ & & \ddots & \\ & & & c(f_n) \end{pmatrix}$$

where f_i divides f_{i+1} and $c(f)$ denotes the companion matrix of f.

Q 7.

Prove that any Hermitian, positive-definite inner product $< -, - >$ on $\mathbf{C}^n$ satisfies $< e_i, e_j >= a_{ij}$ where the matrix $A =^t X\overline{X}$ for some $X \in GL_n(\mathbf{C})$.