Analysis assignment 26/09/09

1) Let $C([0,1])$ denote the space of complex-valued continuous functions on $[0,1]$. Let $\{f_n\}_{n\geq 1} \subset C([0,1])$ be such that $f_n \to f$ uniformly, i.e., $lim_{n\to\infty}\, sup_{t\in[0,1]}|f_n(t)- f(t)| \to 0$. Show that there exists a $M > 0$ such that $|f(t)| \leq M$ for all n, t.

2) Let $\{f_n\}_{n\geq 1}, \{g_n\}_{n\geq 1}$ be two sequences in $C([0,1])$ such that $f_n \to f$ and $g_n \to g$ uniformly. Show that $f_n g_n \to fg$ uniformly.

3) Let $\{f_n\}_{n\geq 1} \subset C([0,1])$ be such that $|f_n(t)| \leq \frac{1}{2^n}$ for all n, t. Show that $g_n = \sum_1^n f_i$ converges uniformly to a g. Such a g is denoted by $\sum_1^\infty f_n$.

4) Let $f_n(x) = \frac{\sin nx}{\sqrt{n}}$. Show that $f_n(x) \to 0$ for all x but $\lim f_n'(x)$ does not exist for any x.

5) Let $f_n(x) = nx(1-x)^n$ for $x \in [0,1]$. Show that $\{f_n\}_{n\geq 1}$ converges point wise but not uniformly. Show that $\int f_n(x)dx \to 0$.

6) Find a sequence (f_n) of continuous functions on $[0,1]$ and a function f on $[0,1]$ such that $f_n \to f$ pointwise but f is not continuous on $[0,1]$.

7) For $n \geq 1$, define f_n by $f_n(x) = 2n^2x$ if $x \in [0, 1/2n]$ and $f_n(x) = 2n(1-nx)$ if $x \in [1/2n, 1/n]$ and $f_n(x) = 0$ if $x \in [1/n, 1]$. Then show that $f_n \to 0$ pointwise but $\int_0^1 f_n(x)dx \not\to 0$.

8) Let $f(x,y) = x^2 + y^2$ if x and y are rational and $f(x,y) = 0$ if x or y is irrational. Is f differentiable at the origin and does f have continuous partial derivatives at the origin.

9) Let $f(x,y) = \frac{xy}{x^2+y^2}$ for $(x,y) \neq (0,0)$ and let $f((0,0)) = 0$. Show that the partial derivatives of f exists but f is not continuous.

10) Let $T : R^n \to R^n$ be a linear map. Show that there exists a $y \in R^n$ such that $T(x) = x.y$, where . denote the dot product of vectors.