Intermediate value property (IVP): $f : (a, b) \to \mathbb{R}$ has this property (also called Darboux property) if $\forall\ x, y \in (a, b),\ x \leq y$

$$[f(x), f(y)](or[f(y), f(x)]) \subseteq \{f(r) : x \leq r \leq y\}.$$

1) Show that if f' exists, f' has IVP.

2) Give examples of non-continuous functions with IVP.

3) Give examples to show that sum of two functions with IVP can fail to have IVP.

4) A function$f : [a, b] \to \mathbb{R}$ is such that $f(\lambda a(1-\lambda)b) \leq \lambda f(a) + (1-\lambda)f(b)$ for all $\lambda \in [0, 1]$. Is f a convex function? (i.e., $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)(f(y)\ \forall\ x, y \in [a, b], \lambda \in [0, 1])$.

5) Suppose $f : [a, b] \to [c, d]$ is such that $f(\frac{x+y}{2}) \leq \frac{f(x)+f(y)}{2}\ \forall\ x, y \in [a, b]$. Show that f is continuous.

6) Suppose f' is continuous on $[a, b]$ and $\epsilon > 0$. Prove that there exists a $\delta > 0 \ni |\frac{f(t)-f(x)}{t-x} - f'(x)| < \epsilon$ whenever $0 < |t - x| < \delta, x, t \in [a, b]$.

7) Suppose f is twice differentiable on $(0, \infty), f'$ is bounded on $(0, \infty)$ and $f(x) \to 0$ as $x \to \infty$. Prove that $f'(x) \to 0$ as $x \to \infty$.

8) Let $f \colon \mathbb{R} \to \mathbb{R}$ be such that $|f(x) - f(y)| \leq (x-y)^2$ for all $x, y \in \mathbb{R}$. Then show that f is a constant.

9) Let $f \colon \mathbb{R} \to \mathbb{R}$ be such that the third derivative of f exists. Suppose the equation $f(a+h) - f(a) = hf'(a+h\theta)$ holds for all reals a, h and for some real θ (independent of a and h). Then show that f is a polynomial of degree at most two.

10) Let $f \colon \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $f'(t) \neq 1$ for all real t. Then prove that f has at most one fixed point and show by example that f need not have any fixed point. In addition if $|f'|$ is bounded by $c < 1$, then show that f has a unique fixed point.

A function $f \colon (a, b) \to \mathbb{R}$ is called analytic if to each $x \in (a, b)$, there exist $\delta > 0$ such that $f(x+h) = \sum_o^\infty a_r h^r$ for all $|h| < \delta$.

11) Show that analytic functions have derivative of all orders and discuss the converse.

12) If f is an analytic function, then either f is constant or $\{x \in (a, b) \mid f(x) = c\}$ has no limit point in (a, b) for any $c \in \mathbb{R}$.