**NBHM Nurture programme**
**Algebra assignment 3**

**Q 1 (on permutation groups and Sylow theorems).**

(1.) *Let $G$ be a finite simple group and $p||G|$. If $G$ has exactly $n > 1$ $p$-Sylow subgroups, show that $G$ is isomorphic to a subgroup of $A_n$.*

(2.) *Let $p$ be a prime and let $a \geq b$ be positive integers. Consider disjoint sets $S_1, \cdots, S_a$ and call their union $S$. Fix a cyclic permutation of each $S_i$ and let $G$ denote the group of permutations of $S$ generated by these $a$ $p$-cycles. Considering the action of $G$ on the set $T$ of all $pb$-element subsets of $S$, prove that*

$$\binom{pa}{pb} \equiv \binom{a}{b} \text{ modulo } p^2.$$

(3.) (i) *Prove that any element in $A_n$ is a commutator $xyx^{-1}y^{-1}$ in $S_n$ where $x$ is an $n$-cycle.*

(ii) *In $S_{2n+1}$, prove that the cycle $(1\ 2\ \cdots\ 2n+1)$ is expressible as $xyx^{-1}y^{-1}$ where $x$ is a $n+1$-cycle.*

(iii) *In any $S_n$, show that every element is a product of at the most two cycles.*

(iv) *Let $F$ be a finite field (if you don't know what that is, just take $\mathbf{Z}/p\mathbf{Z}$ for a prime $p$). Prove that $Sym(F)$ is generated by the permutations $\sigma : x \mapsto x^{-1}$ for $x \neq 0$; $\sigma(0) = 0$ and $\tau_{a,b} : x \mapsto ax + b$ for $a, b \in F$.*

(4.) (i) *For any $n$, it is well-known that the permutations $\sigma = (1\ 2)$ and $\tau = (1\ 2 \cdots n)$ generate the whole of $S_n$. Prove this. Further, if $p$ is a prime, show that any transposition and any $p$-cycle generate $S_p$.*

(ii) *For general $n$, and for a transposition $\sigma$ and any $n$-cycle $\tau$, find a necessary and sufficient condition for $S_n$ to be generated by $\sigma$ and $\tau$.*

(5.) *Prove that $S_n$ is not isomorphic to a subgroup of $A_{n+1}$ for any $n > 1$.*

(6.) *Let $G$ be a finite group and $p||G|$ a prime.*

(i) *Prove that if $P$ is a $p$-Sylow subgroup of $G$, then $N_G(N_G(P)) = N_G(P)$.*

(ii) *Let $N$ be a normal subgroup of $G$ and let $Q \leq N$ be a $p$-Sylow subgroup of $N$. Then, show that $G = NN_G(Q)$.*

(7.) *Let $G$ be a finite group and $p^r$ be a prime power dividing the order of $G$. Then, prove that there exist subgroups of order $p^r$ in $G$ and that these are $\equiv 1$ mod $p$ in number.*

**Q 2 (on nilpotent groups).**
(1.) *Show that for a finite group $G$, the following are equivalent:*
*(i) $G$ is nilpotent, (ii) every proper subgroup $H$ is properly contained in $N_G(H)$, (iii) all maximal subgroups are normal, (iv) all $p$-Sylow subgroups are normal, (v) elements of coprime order commute and, (vi) $G$ is the direct product of its Sylow subgroups.*
(2.) The Frattini subgroup $\Phi(G)$ of a finite group $G$ is defined to be the intersection of all (proper) maximal subgroups.
(i) *Prove that $\Phi(G)$ is the set of 'nongenerators' of $G$ i.e., those elements which can be dropped from any generating set for $G$.*
(ii) *Show that $\Phi(G)$ is a characteristic subgroup.*
(iii) *Prove that $\Phi(G)$ is nilpotent.*
(iv) *For any $p$-group $G$, prove that $\Phi(G) = [G, G] < G^p >$.*
(v) *For a finite abelian $p$-group $A$, show that $A/\Phi(A)$ is an elementary abelian group of order $p^{d(A)}$ where $d(A)$ is the minimal number of generators needed to generate $A$.*
(3.) (i) *Show that a subgroup of a finitely generated nilpotent group is also finitely generated.*
(ii) *Deduce that a finitely generated nilpotent torsion group is finite.*

**Q 3 (on automorphisms).**
*Let $T$ be an automorphism of prime order $p$ of a finite group $G$ such that $T(x) = x$ if, and only if, $x = 1$.*
(i) *Show that the function $F(g) := g^{-1}T(g)$ is a bijection on $G$.*
(ii) *Prove that for any $g$ in $G$, the product $g\ T(g)\ T^2(g) \dots T^{p-1}(g)$ is the identity.*
(iii) *Prove that $|G|$ is congruent to $1$ modulo $p$.*
(iv) *For any prime $q$ dividing the order of $G$, prove that there is a $q$-Sylow subgroup $Q$ of $G$ such that $T(Q) = Q$.*
(v) *For any prime $q$, prove that there is a unique $q$-Sylow subgroup $Q$ of $G$ such that $T(Q) = Q$.*
(vi) *Let $q$ be a prime. Then, show that the $q$-Sylow subgroup $Q$ fixed by $T$ contains any $q$-subgroup of $G$ fixed by $T$.*
(vii) *For $p = 2$, use (i), (ii) to show that $G$ is abelian.*
(viii) *For $p = 3$, show that $G$ is nilpotent.*

**Q 4 (on finitely generated abelian groups).**
(1.) *Using the invariant factor theorem (another name is the theorem of elementary divisors) or otherwise, prove that any unimodular integral vector* $(a_1, \cdots, a_n)$ *(that is, a vector such that the GCD of the* $a_i$*'s is* 1*) is the first column of a matrix in* $SL(n, \mathbf{Z})$ *for any* $n \geq 2$. *Deduce that* $SL(n, \mathbf{Z})$ *acts transitively on the set of unimodular vectors.*
(2.) *Let* $A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in M(2, \mathbf{Z})$ *be any matrix of trace* 0. *If* $(a, b, c)$ *is unimodular, use the conclusion of the previous problem that this vector can be completed to a matrix in* $SL(3, \mathbf{Z})$, *to prove that* $A = BC - CB$ *for some* $B, C \in M(2, \mathbf{Z})$. *Hence, prove for general* $a, b, c$ *also that* $A$ *can be written as above.*

**Q 5 (combinatorial).**
(1.) Let $G$ be a finite group of order $n$ and let $S$ be any subset of $G$. Show that the set $S^n := \{s_1 s_2 \cdots s_n : s_i \in S\}$ is a subgroup of $G$. Assuming that $S$ generates $G$, show further that $S^n$ is normal in $G$ with $G/S^n$ cyclic.
(2.) *Let* $G$ *be a group such that the set* $S$ *of all elements of* $G$ *of finite order is finite. Then, prove that* $S$ *is a group.* (3.) *Let* $G$ *be any group. An action of* $G$ *on a set* $S$ *is said to be transitive if, for each* $s, t \in S$, *there exists* $g \in G$ *such that* $g.s = t$.
(i) *Denote by* $t_n$, *the number of different transitive actions of* $G$ *on* $\{1, 2, \cdots, n\}$. *Prove that the number* $a_n$ *of subgroups of* $G$ *which have index equal to* $n$ *satisfies* $a_n = t_n/(n-1)!$.
(ii) *If* $h_n = |Hom\ (G, S_n)|$, *the number of homomorphisms from* $G$ *to* $S_n$, *then prove that one has the relation*

$$a_n = h_n/(n-1)! - \sum_{k=1}^{n-1} \frac{h_{n-k}}{(n-k)!} a_k.$$

(iii) *Prove that the number of subgroups of index* $n$ *in* $\mathbf{Z}^2$ *is* $\sigma(n)$, *the sum of the divisors of* $n$.
(iv) *Show that application of (ii) to* $\mathbf{Z}^2$ *yields the identity involving the partition function* $p(n)$ *and the sum-of-divisors function* $\sigma(n)$ *:*

$$np(n) = \sum_{i=1}^{n-1} \sigma(i) p(n-i) + \sigma(n).$$

**Q 6 (Miscellaneous).**
1.) *Consider* $S_n$*, the group of permutations of* $A = \{1, 2, \dots n\}$*. Fix an integer* $k$ *with* $1 < k < n$ *and consider all subsets of* $A$ *of cardinality* $k$*. For such a subset* $B$ *and a permutation* $g$*, define* $sgn(g, B)$ *to be 0 if* $g(B) \neq B$ *and* $sign(g|_B)$ *otherwise. (Recall that sign of a permutation is* $\pm 1$ *depending on the parity of number of transpositions needed to write it.) Show that*

$$\sum_{g \in S_n} \Big( \sum_B sgn(g, B) \Big)^2 = 2n! \ .$$

2.) *Let* $G$ *be a group of cardinality* $2k$*, where* $k$ *is odd. First show that* $G$ *has a subgroup of order 2. Then show that* $G$ *has a subgroup of order* $k$*.*
(Hint: Recall the proof of Cayley's theorem. As done there, realize $G$ as a subgroup of the symmetric group $S_{2k}$ thought of as the group of all permutations of elements of $G$.)
3.) *Let* $A$ *and* $B$ *be subsets of a finite group* $G$*. If* $|A| + |B| > |G|$*, show that* $AB = G$*. Here* $AB = \{ab | a \in A, b \in B\}$*.*
4.) *Let* $f$ *be an automorphism of a finite group* $G$ *and let* $I = \{g \in G | f(g) = g^{-1}\}$*. If* $|I| > 3|G|/4$*, show that* $G$ *is abelian. If* $|I| = 3|G|/4$*, show that* $G$ *has an abelian subgroup of index 2. Examples of such* $G$ *and* $f$*?*