II Assignment in Analysis
Nurture 08-09

1. Let $f : [1, \infty) \to \mathbb{R}$ be such that f'' exists,

$$f(1) = f(2) = 4.$$

If f'' is positive in $[1, 3]$ show that

$$f'(3) \geq 0.$$

Hint: Write Taylors formula for $f(1), f(2), f(3)$.

2. Show that $\lim\limits_{\substack{x\to 0\\ x\geq 0}} \frac{x^q-1}{x-1} = q$ for every rational number q.

3. Let

$$\begin{aligned} f(x) &= e^{1/x^2} \quad \text{if} \quad x \neq 0 \\ &= 0 \qquad\ \ \text{if} \quad x = 0. \end{aligned}$$

Use induction to write the formula for $f^{(n)}(x)$ for $x \neq 0$.

4. Give examples of functions f, g on appropriate domains so that the composition $f \circ g$ is Riemann integrable but one of f or g is not Riemann integrable.

5. Let $f : [0, 1] \to [0, 1]$ be a continuous function. Suppose $\int\limits_0^1 f(x)\, dx = 0$.
Show that $f(x) = 0 \ \ \forall\, x \in [0, 1]$.

6. Let $f \colon [0, 1] \to \mathbb{R}$ be a Riemann-integrable function. Then show that for any t between 0 and $\int_0^1 f(x)dx$ there exists $s \in [0, 1]$ such that $f(s) = \int_0^s f(x)dx$.

7. Let $f \colon [0, 1] \to \mathbb{R}$ be a continuous function. Then show that f is integrable. Are integrable functions continuous.

Let $f \colon \mathbb{R} \to \mathbb{R}$ be a function. Then the support of f, denoted by Supp (f) is defined by Supp $(f) = \overline{\{x \in \mathbb{R} \mid f(x) \neq 0\}}$. If Supp (f) is a compact set in $\mathbb{R}$, then we say that f has compact support.

8. Suppose f is a function with compact support. Can f be infinitely differentiable, if so give examples. If f is infinitely differentiable, can f have talyor series expansion.

9. Suppose f is a continuous real-valued functions on reals with compact support. Prove that $f(x) = \int_0^1 f(x+y)dy$ implies $f = 0$ everywhere.