Liftings of Covariant Representations of W^*-correspondences

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14.08.2010

Outline

Introduction

- For a Hilbert C^*-module $\mathcal{G}$, let $\mathcal{L}(\mathcal{G})$ the set of all adjointable operators on $\mathcal{G}$.
- A $\mathcal{G}$ over a von Neumann algebra $\mathcal{M}$ can be equipped with the σ*-topology* induced by $f(.) = \sum_{n=1}^{\infty} \omega_n(\langle \xi_n, . \rangle)$ where $\sum \|\omega_n\| \|\xi_n\| < \infty$.
- $\mathcal{G}$ is called *self-dual* if $\forall \phi : \mathcal{G} \to \mathcal{M} \quad \exists \xi_\phi \in \mathcal{G}$ so that $\phi(\xi) = \langle \xi_\phi, \xi \rangle, \qquad \xi \in \mathcal{G}$.
- For self-dual $\mathcal{G}$, $\mathcal{L}(\mathcal{G})$ is a von Neumann algebra.
- A W^**-correspondence* $\mathcal{E}$ is a self-dual Hilbert C^*-bimodule over $\mathcal{M}$, where the left action $\varphi : \mathcal{M} \to \mathcal{L}(\mathcal{E})$ is normal.
- $\varphi(a)\eta = a\eta \qquad \forall a \in \mathcal{M}, \eta \in \mathcal{E}$.

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Covariant representations

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A pair (T, σ) *is called a* covariant representation *of* $\mathcal{E}$ *over* $\mathcal{M}$ *on a Hilbert space* $\mathcal{H}$, *if*

(i) $T : \mathcal{E} \to B(\mathcal{H})$ *is a linear map that is continuous (w.r.t.* σ *and ultra weak topology)*

(ii) $\sigma : \mathcal{M} \to B(\mathcal{H})$ *is a normal homomorphism*

(iii) $T(a\xi) = \sigma(a)T(\xi), T(\xi a) = T(\xi)\sigma(a) \qquad \xi \in \mathcal{E}, a \in \mathcal{M}.$

Moreover if (T, σ) satisfies

$$T(\xi)^* T(\eta) = \sigma(\langle \xi, \eta \rangle), \qquad \xi, \eta \in \mathcal{E}$$

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   - Representations
   - Preliminaries
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   - Liftings
3. References

σ-dual

- For $\mathcal{E}$ over $\mathcal{M}$ and a normal $\sigma : \mathcal{M} \to B(\mathcal{H})$ the induced tensor product $\mathcal{E} \otimes_\sigma \mathcal{H}$ is the unique Hilbert space such that:

$$\langle \xi_1 \otimes h_1, \xi_2 \otimes h_2 \rangle = \langle h_1, \sigma(\langle \xi_1, \xi_2 \rangle) h_2 \rangle, \qquad \xi_1, \xi_2 \in \mathcal{E}; h_1, h_2 \in \mathcal{H}.$$

- Define *σ-dual* of $\mathcal{E}$ as

$$\mathcal{E}^\sigma := \{\mu \in B(\mathcal{H}, \mathcal{E} \otimes_\sigma \mathcal{H}) : \mu\sigma(a) = (\varphi(a) \otimes \mathbf{1})\mu \quad \forall a \in \mathcal{M}\}.$$

- Let (T, σ) of $\mathcal{E}$ on $\mathcal{H}$ be such that T is bounded. Then $\tilde{T} : \mathcal{E} \otimes \mathcal{H} \to \mathcal{H}$ can be associated with

$$\tilde{T}(\eta \otimes h) := T(\eta)h, \qquad \eta \in \mathcal{E}, h \in \mathcal{H}.$$

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Lemma

let (T,σ) be a covariant representation of $\mathcal{E}$

(i) *T is completely contractive $\Leftrightarrow \|\tilde{T}\| \leq \mathbf{1}$*

(ii) *(T,σ) is isometric if and only if $\tilde{T}$ is an isometry.*

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Liftings

Definition

Let (C, σ_C) be a contractive covariant representation of $\mathcal{E}$ on $\mathcal{H}_C$. Then a contractive covariant representation (E, σ_E) of $\mathcal{E}$ on a $\mathcal{H}_E \supset \mathcal{H}_C$ is called a contractive lifting *of (C, σ_C) if*

(i) $\sigma_E(a)|_{\mathcal{H}_C} = P_{\mathcal{H}_C}\sigma_E(a)|_{\mathcal{H}_C} = \sigma_C(a) \quad a \in \mathcal{M}$

(ii) $\mathcal{H}_C^{\perp}$ *is invariant w.r.t.* $E(\xi)$ *for all* $\xi \in \mathcal{E}$

(iii) $P_{\mathcal{H}_C}E(\xi)|_{\mathcal{H}_C} = C(\xi)$ *for all* $\xi \in \mathcal{E}$

Set $\mathcal{H}_A := \mathcal{H}_C^{\perp}, A(\xi) := E(\xi)|_{\mathcal{H}_A}$ and $\sigma_A(a) := \sigma_E(a)|_{\mathcal{H}_A}$ for all $\xi \in \mathcal{E}, a \in \mathcal{M}$.

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Let (T, σ) be a completely contractive covariant (c.c.c. for short) representation of $\mathcal{E}$ on $\mathcal{H}$.

An isometric dilation (V, π) *of* (T, σ) *is an isometric covariant representation of* $\mathcal{E}$ *on* $\tilde{\mathcal{H}} \supset \mathcal{H}$ *such that* (V, π) *is a lifting of* (T, σ).

A minimal isometric dilation *(mid) of* (T, σ) *is an isometric dilation* (V, π) *on* $\hat{\mathcal{H}}$ *for which*

$$\hat{\mathcal{H}} = \overline{span}\{V(\xi_1)\dots V(\xi_n)h : h \in \mathcal{H}, \xi_i \in \mathcal{E} \text{ for } i = 1, \dots n\}.$$

- mid is unique up to unitary equivalence.

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Full Fock module

- $\mathcal{E} \otimes \mathcal{E}$ w.r.t.

$$\langle \xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2 \rangle = \langle \eta_1, \varphi(\langle \xi_1, \xi_2 \rangle) \eta_2 \rangle, \qquad \xi_1, \xi_2, \eta_1, \eta_2 \in \mathcal{E}.$$

- *Full Fock module* over $\mathcal{M}$ is

$$\mathcal{F} := \oplus_{n=0}^{\infty} \mathcal{E}^{\otimes^n} \text{ where } \mathcal{E}^{\otimes^0} = \mathcal{M}$$

- For $\xi \in \mathcal{E}$ $\quad L_\xi \eta = \xi \otimes \eta \quad \forall \eta \in \mathcal{E}$
- Define $L \otimes \mathbf{1}_{\mathcal{D}_T} : \mathcal{E} \to B(\mathcal{F} \otimes \mathcal{D}_T)$ by

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Presentation of mid

- Set $D_{*,T} := (\mathbf{1} - \tilde{T}\tilde{T}^*)^{\frac{1}{2}}$ (in $B(\mathcal{H})$) and $D_T := (\mathbf{1} - \tilde{T}^*\tilde{T})^{\frac{1}{2}}$ (in $B(\mathcal{E} \otimes_\sigma \mathcal{H})$).
- Let $\mathcal{D}_{*,T} := \overline{\text{range } D_{*,T}}$ and $\mathcal{D}_T = \overline{\text{range } D_T}$.
- Every c.c.c. representation (T, σ) of $\mathcal{E}$ has a mid (V, π), with the representation Hilbert space:

$$\hat{\mathcal{H}} = \mathcal{H} \oplus \mathcal{F} \otimes_{\sigma_1} \mathcal{D}_T$$

$$V(\xi) = \begin{pmatrix} T(\xi) & 0 & 0 & \dots \\ D_T(\xi \otimes \cdot) & 0 & 0 & \dots \\ 0 & 1 & 0 & \\ 0 & 0 & 1 & \\ & & & \ddots \end{pmatrix}$$

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Intertwining unitary

- Let (E, σ_E) be a contractive lifting of (C, σ_C). Clearly mid (V^C, π_C) is embedded in (V^E, π_E). We introduce a c.c.c. representation (Y, π_Y) on the orthogonal complement $\mathcal{K}$ of the space of mid (V^C, π_C) to encode this.
- Hence we can get a unitary W such that

$$W : \mathcal{H}_E \oplus (\mathcal{F} \otimes \mathcal{D}_E) \to \mathcal{H}_C \oplus (\mathcal{F} \otimes \mathcal{D}_C) \oplus \mathcal{K}$$

$$\hat{V}^E(\xi)W = WV^E(\xi), \quad (\pi_C \oplus \pi_Y)(a)W = W\pi_E(a),$$

$$W|_{\mathcal{H}_C} = \mathbf{1}|_{\mathcal{H}_C}, \text{with} \quad \hat{V}^E(\xi) = V^C(\xi) \oplus Y(\xi)$$

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c.n.c.

Lemma

(E, σ_E) is c.c.c. if and only if (C, σ_C) and (A, σ_A) are c.c.c. and $\exists$ a contraction $\gamma : \mathcal{D}_{*,A} \to \mathcal{D}_C$ such that

$$\tilde{B} = D_{*,A}\gamma^* D_C.$$

(A, σ_A) is called *completely non-coisometric (c.n.c.)*, if $\mathcal{H}_A^1 := \{h \in \mathcal{H}_A : \|(\tilde{A}^n)^* h\| = \|h\| \text{ for all } n \in \mathbb{N}\} = 0$.

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Definition

A completely contractive lifting (E, σ_E) *of* (C, σ_C) *by* (A, σ_A) *is called* reduced *if*

1. γ *is* resolving, *i.e., for* $h \in \mathcal{H}_A$

$$\left(\gamma D_{*,A}(A(\xi))^* h = 0 \text{ for all } \xi \in \mathcal{E}\right) \Rightarrow \left(D_{*,A}(A(\xi))^* h = 0 \text{ for all } \xi \in \mathcal{E}\right), \text{ and}$$

2. (A, σ_A) *is c.n.c.*

Definition

The characteristic function of reduced lifting (E, σ_E) *of* (C, σ_C) *is defined as*

$$M_{C,E} := P_{\mathcal{F}\otimes\mathcal{D}_C} W|_{\mathcal{F}\otimes\mathcal{D}_E}.$$

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$$M_{C,E}(L_\xi \otimes \mathbf{1}_E) = (L_\xi \otimes \mathbf{1}_C)M_{C,E}, \qquad \xi \in \mathcal{E}.$$

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For any c.c.c. representation (C, σ_C) of $\mathcal{E}$, the equivalence classes of characteristic functions are complete invariants for reduced liftings of (C, σ_C) up to unitary equivalence.

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