Stinespring's Theorem for maps on Hilbert C^*- modules

G. Ramesh
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Outline

Completely Positive maps

Let $\mathcal{A}$ and $\mathcal{B}$ be C^* *algebras* and $\phi : \mathcal{A} \to \mathcal{B}$ be a linear map. Then ϕ is called

- positive if $\phi(a) \geq 0$ for all $a \geq 0$ in $\mathcal{A}$
- n-positive if $\phi_n : \mathcal{M}_n(\mathcal{A}) \to \mathcal{M}_n(\mathcal{B})$ given by $\phi_n((a_{ij})) := (\phi(a_{ij}))$ is positive
- completely positive (cp) if ϕ_n is positive for every $n \geq 1$.

Stinespring's theorem

Example

Let $\mathcal{A}$ be a unital C^*-algebra, H and K be Hilbert spaces, $V \in \mathcal{B}(H, K)$ and $\rho : \mathcal{A} \to \mathcal{B}(K)$ be a $*$-homomorphism. Then $\phi(a) = V^* \rho(a) V, \quad a \in \mathcal{A}$, is cp.

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Theorem: (W. F. Stinespring 1955)

Let $\mathcal{A}$ be a unital C^*-algebra and $\phi : \mathcal{A} \xrightarrow{\text{cp}} \mathcal{B}(H)$. Then $\exists$ a Hilbert space K, a unital $*$- homomorphism $\rho : \mathcal{A} \to \mathcal{B}(K)$ and $V \in \mathcal{B}(H, K)$ such that

$$\phi(a) = V^* \rho(a) V, \quad a \in \mathcal{A}.$$

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- If $\overline{\text{span}\{\rho(\mathcal{A})VH\}} = K$, then (ρ, V, K) is called a minimal Stinespring representation
- if (ρ_1, V_1, K_1) and (ρ_2, V_2, K_2) are minimal representaions, then there exists a unitary $U : K_1 \to K_2$ such that $UV_1 = V_2$ and $U\rho_1 = \rho_2 U$.

Hilbert C^*-modules

Let $\mathcal{A}$ be a C^*-algebra. An inner-product $\mathcal{A}$-module is a complex linear space E which is equipped with a compatible right $\mathcal{A}$-module structure, together with a map $\langle \cdot, \cdot \rangle E \times E \to \mathcal{A}$ such that; for all $x, y, z \in E,\ \alpha, \beta \in \mathbb{C}$ and $a \in \mathcal{A}$

1. $\langle x, \alpha y + \beta z \rangle = \alpha \langle x, y \rangle + \beta \langle x, z \rangle$
2. $\langle x, ya \rangle = \langle x, y \rangle\, a$
3. $\langle x, y \rangle = \langle y, x \rangle^*$
4. $\langle x, x \rangle \geq 0$
5. $\langle x, x \rangle = 0 \ \Leftrightarrow\ x = 0.$

For $x \in E,\ \|x\| := \|\langle x, x \rangle\|^{\frac{1}{2}}$ is norm on E. If E is complete w.r.t $\|\cdot\|$, then E is called a Hilbert $\mathcal{A}$-module.

Examples

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- $\mathcal{B}(H_1, H_2)$ is Hilbert $\mathcal{B}(H_1)$-module w. r.t the norm induced by $\langle T, S \rangle = T^* S$ for all $T, S \in \mathcal{B}(H_1, H_2)$

Definition

Let $\phi : \mathcal{A} \to \mathcal{B}(H)$ be linear. A map $\Phi : E \to \mathcal{B}(H_1, H_2)$ is said to be a

- ϕ-map if $\langle \Phi(x), \Phi(y) \rangle = \phi(\langle x, y \rangle)$ for all $x, y \in E$;
- ϕ-morphism if Φ is a ϕ-map and ϕ is a morphism;
- ϕ-representation if Φ is a ϕ-morphism and ϕ is a representation.

Theorem: (Mohammed B. Asadi 2009)

Let E be a Hilbert $\mathcal{A}$-module. Let $\phi : \mathcal{A} \xrightarrow{\text{cp}} \mathcal{B}(H_1)$ and $\Phi : E \xrightarrow{\phi-\text{map}} \mathcal{B}(H_1, H_2)$. Assume that

1. $\phi(1) = 1$
2. $\Phi(x_0)\Phi(x_0)^* = I_{H_2}$ for some $x_0 \in E$,

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Then there exist

1. Hilbert spaces K_1, K_2
2. isometries $V : H_1 \to K_1$ and $W : H_2 \to K_2$
3. a $*$-homomorphism $\rho : \mathcal{A} \to \mathcal{B}(K_1)$
4. a ρ-representation $\Psi : E \to \mathcal{B}(K_1, K_2)$

such that

$$\phi(a) = V^*\rho(a)V, \quad \Phi(x) = W^*\Psi(x)V, \quad \text{for all } x \in E, \; a \in \mathcal{A}.$$

Theorem: (B. V. R. Bhat, G. Ramesh & K. Sumesh 2010)

Let $\mathcal{A}$ be a unital C^*-algebra and $\phi : \mathcal{A} \to \mathcal{B}(H_1)$ be a cp map. Let E be a Hilbert $\mathcal{A}$-module and $\Phi : E \to \mathcal{B}(H_1, H_2)$ be a ϕ-map. Then $\exists$ a pair of triples $\big((\rho, V, K_1), (\Psi, W, K_2)\big)$, where

1. K_1 and K_2 are Hilbert spaces;
2. $\rho : \mathcal{A} \to \mathcal{B}(K_1)$ is a unital $*$-homomorphism & $\Psi : E \to \mathcal{B}(K_1, K_2)$ is a ρ-morphism;
3. $V \in \mathcal{B}(H_1, K_1)$ & $W \in \mathcal{B}(H_2, K_2)$

such that

$\phi(a) = V^* \rho(a) V, \;$ for all $\; a \in \mathcal{A}$ and

$\Phi(x) = W^* \Psi(x) V, \;$ for all $\; x \in E$.

Proof

Step I: Existence of (ρ, V, K_1)

Follows from the Stinespring theorem

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Follows from the Stinespring theorem

Step II: Existence of (Ψ, W, K_2)

- $K_2 := \overline{\text{span}\{\Phi(E)H_1\}}$
- For $x \in E$, define $\Psi(x) : K_1 \to K_2$ by
  $\Psi(x)\left(\sum_{j=1}^{n} \rho(a_j)Vh_j\right) := \sum_{j=1}^{n} \Phi(xa_j)h_j$

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- $W := P_{K_2}$ (The orthogonal projection onto K_2)

Definition

Let ϕ and Φ be as above. We say that a pair of triples $((\rho, V, K_1), (\Psi, W, K_2))$ is a Stinespring representation for (ϕ, Φ) if the conditions (1)-(3) of previous Theorem are satisfied. Such a representation is said to be minimal if

1. $K_1 = \overline{\text{span}\{\rho(A)VH_1\}}$
2. $K_2 = \overline{\text{span}\{\Psi(E)VH_1\}}$.

Uniqueness of minimal representations

Let ϕ and Φ be as above. Let $\big((\rho, V, K_1), (\Psi, W, K_2)\big)$ and $\big((\rho', V', K_1'), (\Psi', W', K_2')\big)$ be minimal representations for (ϕ, Φ). Then $\exists$ unitary operators $U_1 : K_1 \to K_1'$ and $U_2 : K_2 \to K_2'$ such that

1. $U_1 V = V', \; U_1 \rho(a) = \rho'(a) U_1, \quad$ for all $a \in \mathcal{A}$ and
2. $U_2 W = W', \; U_2 \Psi(x) = \Psi'(x) U_1, \quad$ for all $x \in E$.

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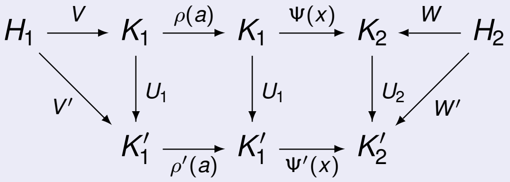
2. $U_2 W = W', \; U_2 \Psi(x) = \Psi'(x) U_1, \quad$ for all $x \in E$.

That is, the following diagram commutes, for $a \in \mathcal{A}$ and $x \in E$:



Proof

Define

- $U_1 : \text{span}(\rho(\mathcal{A})VH_1) \to \text{span}(\rho'(\mathcal{A})V'H_1)$ by

$$U_1\left(\sum_{j=1}^{n} \rho(a_j)Vh_j\right) := \sum_{j=1}^{n} \rho'(a_j)V'h_j,$$

PROOF

Define

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$$U_1(\sum_{j=1}^{n} \rho(a_j)Vh_j) := \sum_{j=1}^{n} \rho'(a_j)V'h_j,$$

- $U_2 : \text{span}(\Psi(E)VH_1) \to \text{span}(\Psi'(E)V'H_1)$ by

$$U_2(\sum_{j=1}^{n} \Psi(x_j)Vh_j) := \sum_{j=1}^{n} \Psi'(x_j)V'h_j.$$

Example I

Let $\mathcal{A} = \mathcal{M}_2(\mathbb{C}),\ H_1 = \mathbb{C}^2, H_2 = \mathbb{C}^8$ and $E = \mathcal{A} \oplus \mathcal{A}$.
Let $D = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}$.

Define $\phi : \mathcal{A} \to \mathcal{B}(H_1)$ by
$\phi(A) = D \circ A,$ for all $A \in \mathcal{A}$ ($\circ$ denote the Schur product).

Let $D_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$ and $D_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}$.

Example II

Let $K_1 = \mathbb{C}^4$ and $K_2 = H_2$.
Define $\Phi : E \to \mathcal{B}(H_1, H_2)$ and $\Psi : E \to \mathcal{B}(K_1, K_2)$ by

$$\Phi(A_1 \oplus A_2) = \begin{pmatrix} \frac{\sqrt{3}}{\sqrt{2}} A_1 D_1 \\ \frac{\sqrt{3}}{\sqrt{2}} A_2 D_1 \\ \frac{1}{\sqrt{2}} A_1 D_2 \\ \frac{1}{\sqrt{2}} A_2 D_2 \end{pmatrix}, \Psi(A_1 \oplus A_2) = \begin{pmatrix} A_1 & 0 \\ A_2 & 0 \\ 0 & A_1 \\ 0 & A_2 \end{pmatrix}, \ A_1, A_2 \in \mathcal{A}.$$

Φ is a ϕ-map.

Example III

Define $V : H_1 \rightarrow K_1$ and $\rho : \mathcal{A} \rightarrow \mathcal{B}(K_1)$ by

$$V = \begin{pmatrix} \frac{\sqrt{3}}{\sqrt{2}} D_1 \\ \frac{1}{\sqrt{2}} D_2 \end{pmatrix}, \quad \rho(A) = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}, \text{ for all } A \in \mathcal{A}.$$

- Ψ is a ρ-morphism
- $\Phi(A_1 \oplus A_1) = W^* \Psi(A_1 \oplus A_2) V$, where $W = I_{H_2}$.

There does not exists an $x_0 \in E$ with the property that $\Phi(x_0)\Phi(x_0)^* = I_{H_2}$, which is an assumption in Asadi's Theorem.

Theorem: Michael Skeide 2010

Let E and F be Hilbert modules over unital C^*-algebras $\mathcal{B}$ and $\mathcal{C}$, respectively. Then for every linear map $T : E \to F$ the following conditions are equivalent:

1. T is a φ-map for some completely positive map $\varphi : \mathcal{B} \to \mathcal{C}$.
2. There exists a pair $(\mathcal{F}, \zeta)$ of a C^*-correspondence F from B to C and a vector $\zeta \in \mathcal{F}$, and there exists an isometry $v : E \odot F \to F$ such that $T = v(id_E \odot \zeta) : x \mapsto v(x \odot \zeta)$.

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THANK YOU

4 Appendix

Even More Additonal Material

Details

text omitted in main Talk

More Details