Measure theoretic quantum white noise calculus

Wilhelm von Waldenfels
Universität Heidelberg

Our treatment of quantum stochastic processes is regarding creation and annihilation operators the dual of Maasen-Meyer kernels. The number operator is the product of a creation operator with an annihilator one. As analytical tool we have available all the instrumentarium of classical measure theory.

Creation and Annihilation Operators

The basic relations of the quantum white noise calculus are the commutation relations

$$[a(s), a^{+}(t)] = \delta(s-t)$$
$$[a(s), a(t)] = [a^{+}(s), a^{+}/t) = 0$$

For the expression $\delta(s-t)$ we have a problem. Whereas the the calculations work perfectly, the mathematical meaning changes with the multiplication of differentials.

If x and y are two *different* real variables we denote the point measure ε_x

$$\varepsilon_x(dy) = \varepsilon(x, dy) : \int \varepsilon_x(dy) f(y) = f(x).$$

Caution!!! $\varepsilon_x(dx)$ is NONSENSE ,not defined.

$$\delta(x-y)dx = \varepsilon_x(dy), \text{ so } x \mapsto \varepsilon_x(dy) \text{ is a measure valued function}$$
$$\delta(x-y)dy = \varepsilon_y(dx), \text{ so } y \mapsto \varepsilon_y(dx) \text{ is a measure valued function}$$
$$\delta(x-y)dx\,dy = \lambda(dx, dy)\delta(x-y)dy \text{ is a measure on } \mathbb{R}^2$$

with

$$\delta(x-y)dx\,dy = \varepsilon_x(dy)dx = \varepsilon_y(dx)dy = \lambda(dx, dy) :$$
$$\int \lambda(dx, dy) f(x, y) = \int dx f(x, x)$$

Define

$$\mathfrak{R} = \{\emptyset\} + \mathbb{R} + \mathbb{R}^2 + \cdots .$$

$+$ denotes disjoint union.The space $\mathfrak{R}$ is locally compact. Assume a continuous symmetric function f on $\mathfrak{R}$.

Annihilation operator :

$$(a(t)f)(t_1, \cdots, t_n) = f(t, t_1, \cdots, t_n)$$

Creation operator :

$$\begin{aligned}&(a^+(dt)f)(t_1, \cdots, t_n)\\ &\qquad = \varepsilon(t_1, dt)f(t_2, \cdots, t_n) + \cdots + \varepsilon(t_n, dt)f(t_1, \cdots, t_{n-1})\end{aligned}$$

(measure valued continuous symmetric function on $\mathfrak{R}$).

Commutation relation

$$[a(s), a^+(dt)] = \varepsilon(s, dt)$$

Number operator

$$(a^+(dt)a(t)f)(t_1, \cdots, t_n) = \sum_{i=1}^{n} \varepsilon(t_i, dt) f(t_1, \cdots, t_n)$$

$a(t)a^+(dt)$ not allowed, includes terms of the form $\varepsilon(t, dt)$.

Multiplication of point measures

- $\varepsilon(x_1, dx_2)\varepsilon(x_3, dx_4) = \varepsilon_{x_1,x_3}(dx_2, dx_4)$ tensor product

- $\varepsilon(x_1, dx_2)\varepsilon(x_2, dx_3) = E(x_1, dx_2, dx_3)$
  $\int E(x_1, dx_2, dx_3) f(x_2, x_3) = f(x_1, x_1)$ multiplication of a measure in dx_2 with a measure valued function in x_2.

$\varepsilon(x_1, dx_2)\varepsilon(x_2, dx_1)$ not defined as

$$\int_{x_2} \varepsilon(x_1, dx_2)\varepsilon(x_2, dx_1) = \varepsilon(x_1, dx_1) \text{ nonsense!!}$$

For short $\varepsilon(x_1, dx_2) = \varepsilon(1,2)$. Consider

$$\varepsilon(b_1, c_1) \cdots \varepsilon(b_n, c_n),$$

where $b_1, \cdots, b_n$ are all different and $c_1, \cdots, c_n$ are all different and $b_i \neq c_i$. Define a relation of right neighborhood in the set

$$S = \{(b_1, c_1), \cdots, (b_n, c_n)\}$$

by

$$(b, c) \triangleright (b', c') \Longleftrightarrow c = b'.$$

As any pair (b, c) has atmost one right neighbor (b', c') , the oriented graph $(S, \triangleright)$ has as components either chains or circuits.

Chain: $(1,2),(2,3),\cdots,(k-1,k)$

$$\varepsilon(1,2)\varepsilon(2,3)\cdots\varepsilon(k-1,k) = E(1;2,3,\cdots,k)$$

$$\int E(1;2,3,\cdots,k)f(2,3,\cdots,k) = f(1,1,\cdots,1) = f(x_1,\cdots,x_n)$$

Circuit: $(1,2),(2,3),\cdots,(k-1,1)$

$$\varepsilon(1,2)\varepsilon(2,3)\cdots\varepsilon(k-1,1) \Longrightarrow \text{Nonsense}$$

integrate over $x_2,\cdots,x_{k-1}$ and arrive at $\varepsilon(x_1,dx_1)$.

Result: *The product*

$$\varepsilon(b_1,c_1)\cdots\varepsilon(b_n,c_n)$$

can be defined if the graph

$$(\{(b_1,c_1),\cdots,(b_n,c_n)\},\triangleright)$$

is without circuits.

Notations

If $\gamma = \{c_1, \cdots, c_n\}$ and f is a symmetric function on $\mathfrak{R}$ then

$$f(t_\gamma) = f(t_{c_1}, \cdots, t_{c_n})$$

is defined regardless of the order of γ. Skipping the letter t we write $f(t_\gamma) = f(\gamma)$. Similar if μ is a symmetric measure on $\mathfrak{R}$ we write

$$\mu(dt_{c_1}, \cdots, dt_{c_n}) = \mu(dt_\gamma) = \mu(\gamma).$$

We write

$$\int \mu(\gamma) f(\gamma) \Delta\gamma$$
$$= f(\emptyset)\mu(\emptyset) + \sum_{n=1}^{\infty} \frac{1}{n!} \int \cdots \int \mu(dt_1, \cdots, dt_n) f(t_1, \cdots, t_n)$$

with $\Delta\gamma = 1/(\#\gamma)!$
Sum-integral-lemma μ a measure on $\Re^k$ with $\mu(dt_{\alpha_1}, \cdots, dt_{\alpha_k})$ symmetric in any variable $dt_{\alpha_1}, \cdots, dt_{\alpha_k}$, then

$$\int \cdots \int_{\Re^k} \mu(dt_{\alpha_1}, \cdots, dt_{\alpha_k}) \Delta\alpha_1 \cdots \Delta\alpha_k = \int_{\Re} \nu(\beta) \Delta\beta$$

with

$$\nu(\beta) = \sum_{\beta_1 + \beta_2 + \cdots + \beta_k = \beta} \mu(\beta_1, \cdots, \beta_k)$$

We denote by λ the Lebesgue measure $\lambda_\emptyset = 1, \lambda_\gamma = dt_{c_1} \cdots dt_{c_n}$ for $\gamma = \{c_1, \cdots, c_n\}$.

Denote

$$a_\gamma^+ = a^+(dt_{c_1}) \cdots a^+(dt_{c_n}) \qquad a_\gamma = a(t_{c_1}) \cdots a(t_{c_n})$$

Admissible monomials

Denote by Φ the function on $\mathfrak{R}$ given by

$$\Phi(w) = \begin{cases} 1 & \text{for } w = \emptyset \\ 0 & \text{for } w \neq \emptyset \end{cases}$$

and by Ψ the meaure on $\mathfrak{R}$ given by

$$\Psi(f) = f(\emptyset)$$

and extend it to measure valued functions.

We define the *measure valued finite particle vectors* $\Phi_\sigma = a_\sigma^+ \Phi$.

Assume two finite sets σ and τ and a finite set of pairs $S = \{(b_i, c_i)\}$,such that all b_i and all c_i are different and $b_i \neq c_i$. We extend the relation $\triangleright$ to the triple (σ, S, τ): If $s \in \sigma, (b,c) \in S, t \in \tau$, then

$$s \triangleright (b,c) \Leftrightarrow s = b, \qquad\qquad (b,c) \triangleright t \Leftrightarrow c = t.$$

Assume the graph $(\sigma, S, \tau, \triangleright)$ without circuits. Assume $\sigma \cap \tau = \emptyset$ and two sets υ, β such that the sets υ, β and $\sigma \cup \tau \cup \bigcup_i \{b_i, c_i\}$ are pairwise disjoint, then with $\varepsilon_S = \varepsilon(b_1, c_1) \cdots \varepsilon(b_n, c_n)$ we have

$$(a_\sigma^+ a_\tau \Phi_\upsilon)(\beta)\varepsilon_S$$

is a well defined product of point measures.

Consider

$$W = (a(\vartheta_n, c_n), \cdots, a(\vartheta_1, c_1))$$

$$a(\vartheta, c) = \begin{cases} a^+(dt_c) & \text{for } \vartheta = +1 \\ a(t_c) & \text{for } \vartheta = -1 \end{cases}$$

We call W *admissible* if

$$i > j \Longrightarrow \{c_i \neq c_j \text{ or } \{c_i = c_j \text{ and } \vartheta_i = +1, \vartheta_j = -1\}\}.$$

W normal ordered

$$W = (a^+(ds_1), \cdots, a^+(ds_l), a^+(dt_1), \cdots a^+(dt_m), a(t_1), \cdots, a(t_m), \\ a(u_1), \cdots, a(u_n)) = a^+_{\sigma+\tau} a_{\tau+\upsilon}$$

A normal ordered sequence W is admissible, the juxtaposition of two normal ordered sequences is in general not normal ordered, but it is admissible, provided the variables are different.

Cosider an admissible sequence and denote . We consider the set $\mathfrak{P}(W)$ of all decompositions of $[1,n]$, i.e. all sets of subsets of $[1,n]$ of the following form

$$\begin{aligned} \mathfrak{p} &= \{\mathfrak{p}_+, \mathfrak{p}_-, \{q_i, r_i\}_{i\in I}\} \\ [1,n] &= \mathfrak{p}_+ + \mathfrak{p}_- + \sum_{i\in I}\{q_i, r_i\} \\ & \mathfrak{p}_+ \subset \{j : \vartheta_j = 1\}; \mathfrak{p}_- \subset \{j : \vartheta_j = -1\}; \\ & \vartheta_{q_i} = -1,\ \vartheta_{r_i} = 1; q_i > r_i \end{aligned}$$

For $\mathfrak{p} \in \mathfrak{P}(W)$ we define

$$\lfloor W \rfloor_{\mathfrak{p}} = \prod_{s\in\mathfrak{p}_+} a^+_{c_s} \prod_{i\in I} \varepsilon(c_{q_i}, c_{r_i}) \prod_{t\in\mathfrak{p}_-} a_{c_t}$$

The triple $(\mathfrak{p}_+, \bigcup_{i\in I}\{q_i, r_i\}, \mathfrak{p}, \triangleright)$
is without circuits, so the product $\lfloor W \rfloor_{\mathfrak{p}} \Phi_\upsilon$
is well defined for any finite particle vector Φ_υ,
with $\upsilon \cap \{c_1, \cdots, c_n\} = \emptyset$.

The product

$$M = a(\vartheta_n, c_n) \cdots a(\vartheta_1, c_1) \Phi_\upsilon$$

can be defined by successive application.

Wick's Theorm

$$M = \sum_{\mathfrak{p} \in \mathfrak{P}(W)} \lfloor W \rfloor_{\mathfrak{p}}.$$

Example 1. Assume

$$M = a(dt_4)a^+(dt_3)a(dt_2)a^+(dt_1) = a(4)a^+(3)a(2)a^+(1)$$

then

$$\begin{aligned} M = a^+(3)a^+(1)a(4)a(3) + \varepsilon(43)a^+(1)a(2) \\ + \varepsilon(41)a^+(3)a(2) + \varepsilon(21)a^+(3)a(4) + \varepsilon(43)\varepsilon(21) \end{aligned}$$

Example 2.

$$M = a(3)a^+(2)a(2)a^+(1) = \varepsilon(32)\varepsilon(21) = E(3;21)$$

If

$$M = a(\vartheta_n, c_n) \cdots a(\vartheta_1, c_1)$$

is admissible, then denote

$$\Psi M \Phi = \langle M \rangle.$$

If n is odd, then $\langle M \rangle = 0$. If $n = 2m$ is even, denote by $\mathfrak{P}_0(2m)$ the set of all those pair partitions

$$\mathfrak{p} = \{\{p_1, q_1\}, \cdots, \{p_m, q_m\}\}$$

with $p_i > q_i, \vartheta_{p_i} = -1, \vartheta_{q_i} = +1$. Then

$$\langle M \rangle = \sum_{\mathfrak{p} \in \mathfrak{P}_0(2m)} \lfloor M \rfloor_{\mathfrak{p}} = \sum_{\mathfrak{p} \in \mathfrak{P}_0(2m)} \prod_{i=1}^{m} \varepsilon(c_{p_i}, c_{q_i}).$$

Denote

$$\omega = \{c_1, \cdots, c_n\} \quad \omega_+ = \{c_i : \vartheta_i = +1\} \quad \omega_- = \{c_i : \vartheta_i = -1\}$$

For any $\mathfrak{p}$ the graph $(\mathfrak{p}, \triangleright)$ the product of the point measures can be defined. The starting points of the chains are the elements of $\omega_- \setminus \omega_+$. So $\lfloor M \rfloor_{\mathfrak{p}}$ is the tensor product of measures of the form $E(x_1; dx_2, \cdots, dx_k)$ defined above and $\langle M \rangle$ is a continuous function

$$\mathbb{R}^{\omega_- \setminus \omega_+} \to \mathcal{M}_+(\mathbb{R}^{\omega_+}).$$

Now

$$dx_1 E(x_1; dx_2, \cdots, dx_k) = \Lambda(dx_1, \cdots, dx_k)$$
$$\int \Lambda(dx_1, \cdots, dx_k) f(x_1, \cdots, x_k) = \int dx f(x, \cdots, x)$$

Multiply with

$$\lambda^{\omega_- \setminus \omega_+} = \prod_{i \in \omega_- \setminus \omega_+} dx_i$$

and obtain a positive measure on $\mathbb{R}^\omega$.

$$\langle M \rangle \lambda^{\omega_- \setminus \omega_+} = \sum_{\mathfrak{p} \in \mathfrak{P}_0(2m)} \lfloor M \rfloor_{\mathfrak{p}} \lambda^{\omega_- \setminus \omega_+}$$

and any term is the tensor product of measures Λ.

With

$$M^T = a(-\vartheta_1, c_1) \cdots a(-\vartheta_n, c_n)$$

one obtains the symmetry relation

$$\langle M \rangle \lambda^{\omega_- \setminus \omega_+} = \langle M^T \rangle \lambda^{\omega_+ \setminus \omega_-}.$$

Representation of unity If $M = M_2 M_1$ is admissible, then

$$\langle M_2 M_1 \rangle = \int_\alpha \langle M_2 a_\alpha^+ \rangle \langle a_\alpha M_1 \rangle \mathbf{\Delta}\alpha$$

Remark. If M is admissible, and π, ϱ, ω are pairwise disjunkt, then

$$\langle a_\pi M a_\varrho^+ \rangle \lambda^{\pi + \omega_- \backslash \omega_+}$$

is a measure on $\mathfrak{R} \times \mathbb{R}^\omega \times \mathfrak{R}$. The general form of a normal ordered monomial is $a^+_{\sigma+\tau} a_{\tau+\upsilon}$and

$$\langle a_\pi a^+_{\sigma+\tau} a_{\tau+\upsilon} a_\varrho^+ \rangle \lambda^{\pi+\upsilon}$$

defines a measure on $\mathfrak{R}^5$ letting run σ, τ, υ.

Quantum stochastic differential equation

Hudson-Parthasarathy

$$d_t U_s^t = A_1 dB_t^+ U_s^t + A_0 d\Lambda_t U_s^t + A_{-1} dB_t U_s^t + BU_s^t dt,\ U_s^s = 1$$

where A_1, A_0, A_{-1}, B are Operators in $B(\mathfrak{k})$, where $\mathfrak{k}$ Hilbert pace.

Accardi: *normal ordered* equation

$$dU_s^t/dt = A_1 a_t^+ U_s^t + A_0 a_t^+ U_s^t a_t + A_{-1} U_s^t a_t + BU_s^t$$

Our approach is very similar to Accardi's one. We understand U_s^t as a sesquilinear form over $\mathcal{K}_s(\mathfrak{R}, \mathfrak{k})$ (symmetric continuous functions $\mathfrak{R} \to \mathfrak{k}$ of compact support)

$$\langle f|U_s^t|g\rangle =$$
$$\int \cdots \int f^+(\pi) u_s^t(\sigma, \tau, \upsilon) g(\varrho) \langle a_\pi a_{\sigma+\tau}^+ a_{\tau+\upsilon} a_\varrho^+ \rangle \lambda^{\pi+\varrho} \Delta\pi \Delta\varrho \Delta\sigma \Delta\tau \Delta\upsilon$$

where u_s^t is a locally Lebesgue integrable function $\mathbb{R} \times \mathfrak{R}^3 \to B(\mathfrak{k})$in all four variables $t, t_\sigma, t_\tau, t_\upsilon$. We formulate the differential equation in the weak sense

$$(d/dt)\langle f|U_s^t|g\rangle$$
$$= \langle a(t)f|A_1 U_s^t|g\rangle + \langle a(t)f|A_0 U_s^t|a(t)g\rangle + \langle f|A_{-1}U_s^t|a(t)g\rangle + \langle f|BU_s^t|g\rangle$$

or better as integral equation

$$\langle f|U_s^t|g\rangle = \langle f|g\rangle + \int_s^t dr\langle a(r)f|A_1 U_s^r|g\rangle + \int_s^t dr\langle a(r)f|A_0 U_s^r|a(r)g\rangle + \int_s^t dr\langle f|A_{-1}U_s^r|a(r)g\rangle + \int_s^t dr\langle f|BU_s^r|g\rangle$$

for $t \geq s$.

Theorem The equation has a unique solution, given in the following way. Assume that all points $s, t, t_\sigma, t_\tau, t_\upsilon$ are different and order

$$t_\sigma + t_\tau + t_\upsilon = \{s_1 < \cdots < s_n\}$$

and define

$$i_j = \begin{cases} 1 & \text{if } j \in \sigma \\ 0 & \text{if } j \in \tau \\ -1 & \text{if } j \in \upsilon \end{cases}$$

Then

$$\begin{aligned} u_s^t(t_\sigma, t_\tau, t_\upsilon) = \mathbf{1}\{s < s_1 < \cdots < s_n < t\} \\ \exp((t - s_n)B) A_{i_n} \exp((s_n - s_{n-1})B) A_{i_{n-1}} \\ \times \cdots \times A_{i_2} \exp((s_2 - s_1)B) A_{i_1} \exp((s_1 - s)B) \end{aligned}$$

The solution has a remarkable easy analytical structure. Assume a function

$$x : (t, t_\sigma, t_\tau, t_\upsilon) \in \mathbb{R} \times \mathfrak{R}^k \mapsto x_t(t_\sigma, t_\tau, t_\upsilon) \in B(\mathfrak{k})$$

symmetric in $t_\sigma, t_\tau, t_\upsilon$. Then x is called of class $\mathcal{C}^0$ if the function is locally integrable and is continuous in the subspace, where all points $t, t_\sigma, t_\tau, t_\upsilon$ are different. We call x of class $\mathcal{C}^1$ if it is of class $\mathcal{C}^0$ and if on the same subspace the functions

$$\begin{aligned}
\partial^c x_t(t_\sigma, t_\tau, t_\upsilon) &= (d/dt) x_t(t_s, t_\tau, t_\upsilon) \\
(R^1_\pm x)_t(t_\sigma, t_\tau, t_\upsilon) &= x_{t\pm 0}(t_\sigma + \{t\}, t_\tau, t_\upsilon) \\
(R^0_\pm x)_t(t_\sigma, t_\tau, t_\upsilon) &= x_{t\pm 0}(t_\sigma, t_\tau + \{t\}, t_\upsilon) \\
(R^{-1}_\pm x)_t(t_\sigma, t_\tau, t_\upsilon) &= x_{t\pm 0}(t_\sigma, t_\tau, t_\upsilon + \{t\}) \\
(D^i x)_t &= (R^i_+ x)_t - (R^i_- x)_t
\end{aligned}$$

exist and are of class $\mathcal{C}^0$. *The solution of the quantum stochastic differential equation is of class* $\mathcal{C}^1$

Ito's Theorem Assume $F, G : \mathfrak{R}^3 \to B(\mathfrak{k})$ to be λ-measurable and define the sesquilinear form over $\mathcal{K}s(\mathfrak{R}, \mathfrak{k})$

$$\langle f|\mathcal{B}(F,G)|g\rangle = \int \cdots \int \langle a_\pi a^+_{\sigma_1+\tau_1} a_{\tau_1+\upsilon_1} a^+_{\sigma_2+\tau_2} a_{t_2+\upsilon_2} a^+_\varrho \rangle \lambda_{\pi+\upsilon_1+\upsilon_2}$$
$$\Delta\pi \cdots \Delta\upsilon_2 f^+(\pi) F(\sigma_1,\tau_1,\upsilon_1) G(\sigma_2,\tau_2,\upsilon_2) g(\varrho)$$

provided the integral exists in norm. Assume x_t, y_t to be of class $\mathcal{C}^1$ and that for $f, g \in \mathcal{K}_s(\mathfrak{R}, \mathfrak{k})$ the sesquilinear forms $\langle f|\mathcal{B}(F_t, G_t)|g\rangle$ exist in norm and $t \in \mathbb{R} \mapsto \langle f|\mathcal{B}(F_t, G_t)|g\rangle$ is locally integrable, where

$$F_t \in \{x_t, \partial^c x_t, R^1_\pm x_t, R^0_\pm x_t, R^{-1}_\pm x_t\}$$

$$G_t \in \{y_t, \partial^c y_t, R^1_\pm y_t, R^0_\pm y_t, R^{-1}_\pm y_t\}$$

is any of the functions.

Then the Schwartz derivative of $\langle f|\mathcal{B}(x_t, y_t)|g\rangle$ is a locally integrable function and yields

$$\begin{aligned}\partial\langle f|\mathcal{B}(x_t, y_t)|g\rangle =& \langle f|\mathcal{B}(\partial^c x_t, y_t) + \mathcal{B}(x_t, \partial^c y_t) + I_{-1,+1,t}|g\rangle \\ &+ \langle a(t)f|\mathcal{B}(D^1 x_t, y_t) + \mathcal{B}(x_t, D^1 y_t) + I_{0,+1,t}|g\rangle \\ &+ \langle a(t)f|\mathcal{B}(D^0 x_t, y_t) + \mathcal{B}(x_t, D^0 y_t) + I_{0,0,t}|a(t)g\rangle \\ &+ \langle f|\mathcal{B}(D^{-1} x_t, y_t) + \mathcal{B}(x_t, D^{-1} y_t) + I_{-1,0,t}|a(t)g\rangle\end{aligned}$$

with

$$I_{i,j,t} = \mathcal{B}(R^i_+ x_t, R^j_+ y_t) - \mathcal{B}(R^i_- x_t, R^j_- y_t).$$

We define the Fock space

$$\Gamma = L_s^2(\mathfrak{R}, \mathfrak{k}, \lambda)$$

of all symmetric square integrable functions with respect to Lebesgue measure from $\mathfrak{R}$ to $\mathfrak{k}$. If f is a mesurable function on $\mathfrak{R}$ define the operator N by $(Nf)(w) = (\# w)f(w)$ and define Γ_k as the space of those measurable symmetric functions from $\mathfrak{R}$ to $\mathfrak{k}$, for which

$$\int \langle f(w) | (N+1)^k f(w) \rangle \, dw < \infty$$

We denote by $\|.\|_{\Gamma_k}$ the corresponding norm.

Unitarity

There exists a family of *unitary* operators $\tilde{U}_s^t : \Gamma \to \Gamma$ such that

$$\langle f|\tilde{U}_s^t|g\rangle = \langle f U_s^t|g\rangle$$

for $f, g \in \mathcal{K}_s(\mathfrak{R}, \mathfrak{k})$ iff the operators $A_i, i = 1, 0, -1; B$ fulfill the following conditions: There exist a unitary operator Υ such that

$$\begin{aligned} A_0 &= \Upsilon - 1 \\ A_1 &= -\Upsilon A_{-1}^+ \\ B + B^+ &= -A_1^+ A_1 = -A_{-1} A_{-1}^+. \end{aligned}$$

We write $U_s^t = \tilde{U}_s^t$. Furthermore there exists a polynomial P of degree $\leq k$,such that for $f \in \Gamma_k$

$$\|U_s^t f\|_{\Gamma_k} \leq P(|t-s|)\|f\|_{\Gamma_k}$$

$$\|U_s^t f - f\|_{\Gamma_k} \to 0$$

Characterization of the Hamiltonian Define for $t < 0$ the operator $U_0^t = (U_t^0)^+$ and denote by $\Theta(t)$ the right shift on $\mathfrak{R}$.Then

$$t \to W(t) = \Theta(t) U_0^t$$

ist a strongly continuous unitary one paremeter group on Γ. By Stone's theorem there exists a closed selfadjoint operator H with dense domain $D_H \subset \Gamma$ such that

$$W(t) = e^{-iHt}.$$

We want to give an explicit representation of H.(Accardi, Chebotariew, Belavkin, Gregoratti)

If $\varphi \in (L^1 \cap L^2)(\mathbb{R})$ define $\Theta(\varphi) = \int \varphi(t)\Theta(t)dt$. Denote for $f \in L^2(\mathbb{R}^n)$

$$\begin{aligned}\mathfrak{a} = a(0) :& (\mathfrak{a}f)(t_2, \cdots, t_n) = f(0, t_2, \cdots, t_n) \\ \mathfrak{a}^+ :& (\mathfrak{a}^+ f)(dt_0, \cdots, dt_n) = \varepsilon_0(dt_0) f(t_1, t_2, \cdots, t_n) dt_1 \cdots dt_n) \\ & + \cdots + \varepsilon_0(dt_n) f(t_0, \cdots, t_{n-1}) dt_0 \cdots dt_{n-1}\end{aligned}$$

The operator $\Theta(\varphi)$ works as mollifyer and makes out of the singular measure $\mathfrak{a}^+ f$ a measure with density, wich we identify with its density

$$(\Theta(\varphi)\mathfrak{a}^+ f)(t_0, \cdots, t_n) =$$
$$\varphi(-t_0)(\Theta(-t_0)f)(t_1, \cdots, t_n) + \cdots + \varphi(-t_n)(\Theta(-t_n)f)(t_0, \cdots, t_{n-1}).$$

We double the point 0 and introduce

$$\mathbb{R}_0 =]-\infty, 0] + [0, \infty]$$
$$\mathfrak{R}_0 = \{\emptyset\} + \mathbb{R}_0 + \mathbb{R}_0^2 + \cdots$$

We have the point measures $\varepsilon_{\pm 0}$, define accordingly $\mathfrak{a}_\pm, \mathfrak{a}_\pm^+$ and

$$\hat{\mathfrak{a}} = \frac{1}{2}(\mathfrak{a}_+ + \mathfrak{a}_-) \qquad \hat{\mathfrak{a}}^+ = \frac{1}{2}(\mathfrak{a}_+^+ + \mathfrak{a}_-^+)$$

We call a δ-sequence φ_n , i.e. $\varphi_n \to \delta$ a *symmetric* δ-sequence if the φ_n are real and if $\varphi_n(t) = \varphi_n(-t)$ for all n and t. We define the *symmetric* differentiation $\widehat{\partial}$ by

$$\widehat{\partial} = -\lim \Theta(\varphi_n'),$$

where φ_n is a symmetric δ-sequence.

Define

$$Z = \int_0^\infty e^{-t}\Theta(t)dt$$

$$D = \{f = Z(f_0 + \mathfrak{a}^+ f_1) : f_0 \in \Gamma_1, f_1 \in \Gamma_2\}$$

The sesquilinear form

$$f, g \in D \mapsto \langle f | i\widehat{\partial} g\rangle = \int f^+(\omega)(i\widehat{\partial} g)(\omega)\lambda_\omega$$

exists and is symmetric,i.e.$\langle f | i\widehat{\partial} g\rangle = \langle i\widehat{\partial} f | g\rangle$. Assume four operators $M_0, M_{\pm 1}, G \in B(\mathfrak{k})$ such that

$$M_0^+ = M_0 \qquad M_1^+ = M_{-1} \qquad G^+ = G,$$

then define by

$$\widehat{H} = i\widehat{\partial} + M_1\widehat{\mathfrak{a}}^+ + M_0\widehat{\mathfrak{a}}^+\widehat{\mathfrak{a}} + M_{-1}\widehat{\mathfrak{a}} + G.$$

an application from D into the singular measures on $\mathfrak{R}_0$. The operator is symmetric, the singular part of $\hat{H}f$ is given by

$$\hat{\mathfrak{a}}^{+}(-if_1 + M_1 f + M_0 \hat{\mathfrak{a}} f).$$

Denote by D_0 the subspace, where the singular part vanishes and by H_0 the restriction of $\hat{H}$ to D_0.

Theorem Assume

$$A_1 = \frac{1}{i - M_0/2} M_1$$
$$A_0 = \frac{M_0}{i - M_0/2}$$
$$A_{-1} = M_{-1} \frac{1}{i - M_0/2}$$
$$B = -iG - \tfrac{i}{2} M_{-1} \frac{1}{i - M_0/2} M_1$$

Then the domain D_H of the Hamiltonian H of $W(t)$ contains D_0 and the restriction of H to D_0 coincides with the restriction H_0 of

$$\widehat{H} = i\widehat{\partial} + M_1 \widehat{\mathfrak{a}}^+ + M_0 \widehat{\mathfrak{a}}^+ \widehat{\mathfrak{a}} + M_{-1} \widehat{\mathfrak{a}} + G.$$

to D_0 and D_0 is dense in Γ and H is the closure of H_0.