

# The Lambert W Function

$$W(z)e^{W(z)} = z$$

$$ye^y = z \iff y = W_k(z)$$

# D-dimensional Bose Gases and the W Function

$$z^{z^{z^{\dots}}} = \frac{W(-\ln z)}{-\ln z}$$

### A Fractal Related to W

Each colour represents a cycle length in the iteration  $z \mapsto z^{z^{\dots}}$  with  $z = x + iy$ . A point at coordinate  $(x, y)$  is by where  $C = 1.04714$  is given the colour corresponding to the length of its orbiting cycle.

$$W(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{k+1}}{(k+1)!}$$

$$\frac{d}{dz} W(z) = \frac{W(z)}{z(1+W(z))}$$

if  $z \neq 0, -1/e$

### Johann Heinrich Lambert

Johann Heinrich Lambert was born in Mulfingen on the 25th of August, 1728, and died in Berlin on the 25th of September, 1777. His scientific interests were remarkably broad. The self-educated son of a tailor, he produced fundamentally important work in number theory, geometry, statistics, astronomy, meteorology, hygrometry, pyrometry, optics, cosmology and philosophy. Lambert was the first to prove the irrationality of  $\pi$ . He worked on the parallel postulate, and also introduced the modern notation for the hyperbolic functions.

In a paper entitled "Observationes Variae in Mathesei Puram", published in 1758 in Acta Helvetica, he gave a series solution of the trinomial equation,  $x^{2m} + px - q = 0$ , for  $x$ . His method was a precursor of the more general Lagrange inversion theorem. This solution intrigued his contemporary, Euler, and led to the discovery of the Lambert W function.

Lambert wrote Euler a cordial letter on the 18th of October, 1771, expressing his hope that Euler would regain his sight after an operation; he explains in this letter how his trinomial method extends to series reversion.

The Lambert W function is implicitly elementary. That is, it is implicitly defined by an equation containing only elementary functions. The Lambert W function is not, itself, an elementary function; it is not a Liouvillian function, which means that it is not expressible as a finite combination of algebraic, root, logarithmic, or antiderivatives (quadratures) of any other areas.

### Leonhard Euler



Leonhard Euler was born on the 15th of April, 1707, in Basel, Switzerland, and died on the 18th of September, 1803, in St. Petersburg, Russia. His papers were written in the last fourteen years of his life, even though he had gone blind.

Euler was the greatest mathematician of the 18th century, and one of the greatest of all time. His work on the calculus of variations has been called "the most beautiful book ever written", and Pierre Simon de Laplace referred his students: "Lisez Euler, c'est votre maître et l'art". Advice that is also profitable today.

Many functions and concepts are named after him, including the Euler totient function, Eulerian numbers, the Euler-Lagrange equations, and the "Eulerian" formulation of fluid mechanics. The mathematical formulae on this poster are the Euler formula, designed by Hermann Zapf to evoke the flavour of excellent human handwriting.

Lambert's series solution of his trinomial equation, which Euler reverts as  $x^{2m} + px - q = (x - \beta)(x^{2m} + \beta x^{2m-1} + \dots + \beta^m)$ , led to the series solution of the transcendental equation  $x e^{x^2} = y$ . This was the earliest known occurrence of the series for the function now called the Lambert W function.

### Applications of W



Hippasus of Eleus lived, travelled and worked around 490 BC, and is mentioned by Plato. The Quadratrix (or trisectrix) of Hippasus is the first curve ever mentioned by name. As drawn in the picture here, its equation is  $y = x \tan(x)$ . This curve can be used to square the circle and to trisect the angle. Since these classical problems are unsolvable by straightedge and compass, we therefore consider that the construction of the Quadratrix is impossible under that restriction. The Quadratrix is also the image of the real axis under the map  $z \mapsto W_k(z)$ , and the points of the curve corresponding to the negative real axis define the ranges of the branches of W with different colours.

### Sir Edward Maitland Wright



Sir Edward Maitland Wright was born the 1st of January, 1906. He is the co-author with G. H. Hardy of the classic book An Introduction to the Theory of Numbers. His main contributions to the study of the Lambert W function were a systematic way of computing complex values, a series expansion of a related function about its branch points, the application of W to enumeration problems, and the application of W to the study of the stability of the solutions of some nonlinear delay differential equations. He was Professor of Mathematics, then Principal and Vice-Chancellor, of Aberdeen University (1959-1976).

$$z = \ln e^z + 2\pi i K(z)$$

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$$\int W(z) dz = \frac{z(W^2(z) - W(z))}{W(z)}$$

$$\int_0^{\infty} x^{s-1} W(x) dx = \frac{(-s)^{-1} \Gamma(s)}{s} \quad \text{if } -1 < \text{Re}(s) < 0$$

$$\int 2 \sin W(x) dx = \left(x + \frac{x}{W(x)}\right) \sin W(x) - x \cos W(x) + C$$

$$\int_0^{\infty} e^{-st} W(e^t) dt = s^{-2} \Gamma(1-s, sW(1)) + \frac{W(1)}{s} \quad \text{if } \text{Re}(s) > 0$$

R.M. Corless, G.H. Gonnet, D.E.G. Hare, D.J. Jeffrey, and D.E. Knuth.  
"On the Lambert W Function", Advances in Computational Mathematics, volume 5, 1996, pp. 329-359  
[www.orcca.on.ca/LambertW](http://www.orcca.on.ca/LambertW)

# Collaborators

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1. The Lambert  $W$  function and quantum statistics, *Journal of Mathematical Physics*, **50**, 2009
2. D-Dimensional Bose gases and the Lambert  $W$  function  
accepted with minor revisions, February 2010

# Introduction

- The applications of the  $W$  function (hereafter mostly referred to as the  $W$  function) to  $D$ -dimensional Bose gases are presented in this talk.
- The low temperature  $T$  behavior of free ideal Bose gases is considered in 2, 3 and 4 dimensions.

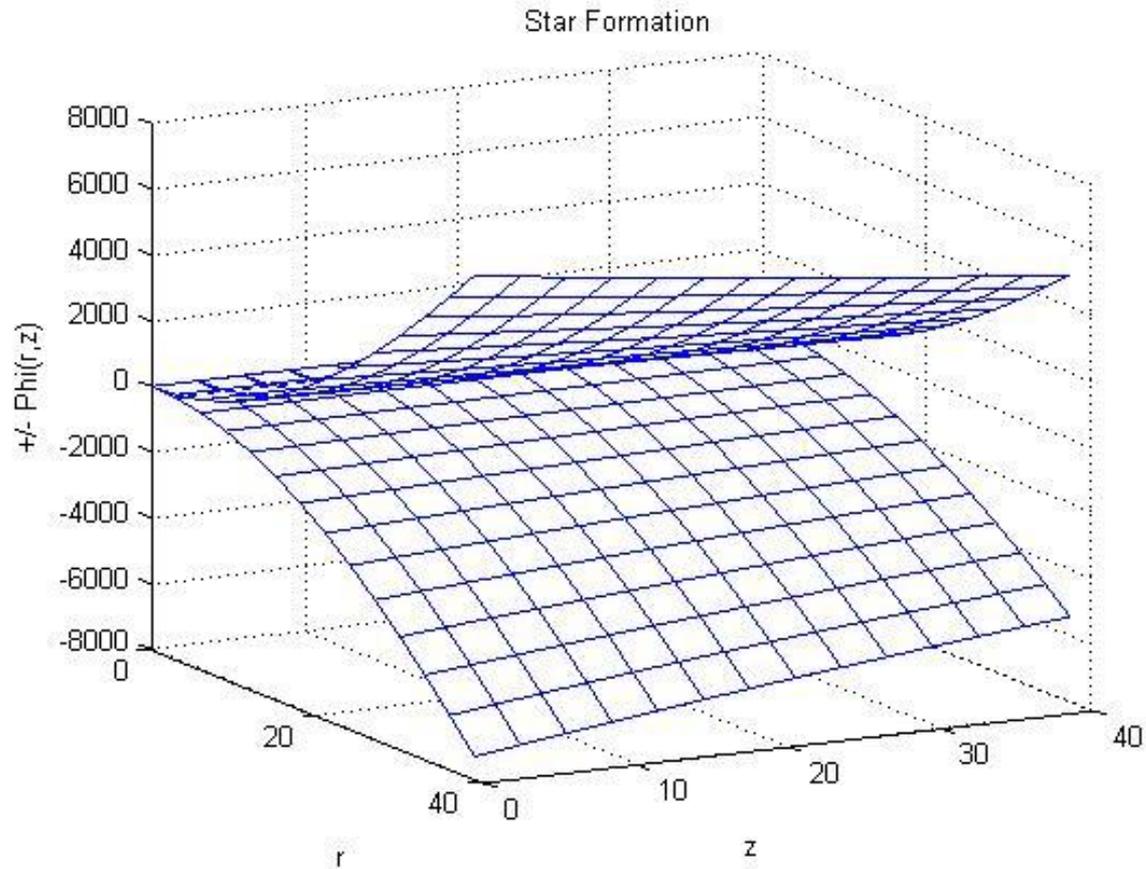
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# 1. Applications of the $W$ function

- Enumerating search trees
- Solutions to transcendental equations
- Analytic solutions to solar cell parameters
- Wien's displacement law
- Fringing fields of a parallel plate capacitor

The  **$W$  function** has the potential to provide solutions to problems that have previously not been solved analytically, as well as uncovering novel and interesting properties of previously solved problems.



A function  $\Phi(r,z)$  that can be used to generate magnetic field lines in terms of the Lambert  $W$

## 2. The W function

The W function is defined as the multivalued function which solves the following equation:

$$W(z) \exp(W(z)) = z \quad z \in \mathbb{C}, \quad (1)$$

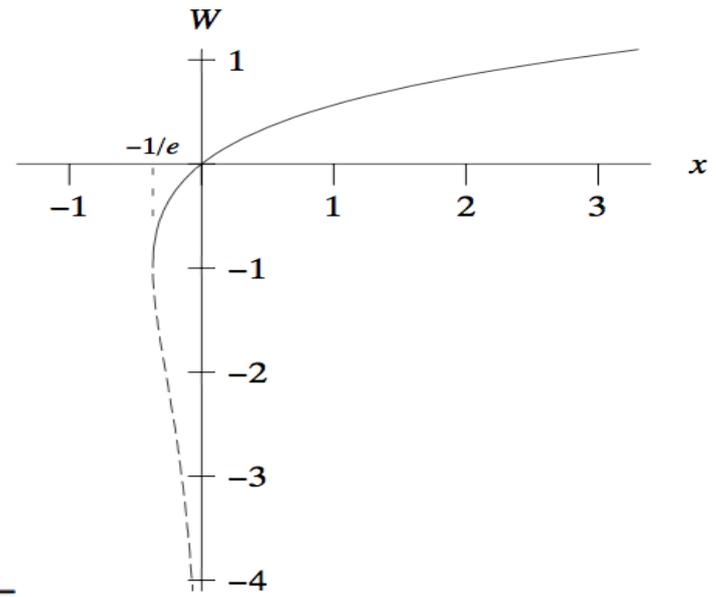
or, equivalently, as the multivalued inverse of the function

$$f : z \rightarrow ze^z$$

For real argument, at most two solutions:

- $W_0$  (solid line) is the principal branch
- $W_{-1}$  (dashed line)

$$W_0(z) = \sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!} z^n = z - z^2 + \frac{3}{2}z^3 - \dots$$



(2)

Two families of transcendental equations appear in statistical mechanics, applied mathematical and physics problems.

Type 1: 
$$x^n + B \ln x + C = 0 . \quad (3)$$

where  $n, B, C \in \mathbb{C}$  not depending on  $x$  and  $n, B \neq 0$ . Then

$$\frac{n}{B}x^n + n \ln x = -\frac{nC}{B} \Rightarrow \frac{n}{B}x^n e^{\frac{n}{B}x^n} = \frac{n}{B}e^{-\frac{nC}{B}} .$$

Type 2: 
$$x = \left[ \frac{B}{n} W_j \left( \frac{n}{B} \exp \left( -\frac{nC}{B} \right) \right) \right]^{1/n} . \quad (4)$$

$$x^n \ln x + Bx^n + C = 0 . \quad (5)$$

$$n \ln x + nB = -Cnx^{-n} \Rightarrow -Cne^{nB} = -Cnx^{-n} e^{-Cnx^{-n}} .$$

$$x = \left[ -\frac{1}{Cn} W_j (-Cne^{nB}) \right]^{-1/n} . \quad (6)$$

### 3A. Thermodynamic Functions

$$\bar{n}_{BE} = -\frac{1}{Z_{BE}} \frac{\partial Z_{BE}}{\partial x} = \frac{1}{e^{\beta(\epsilon-\mu)} - 1}, \quad (7)$$

$$\bar{n}_{FD} = -\frac{1}{Z_{FD}} \frac{\partial Z_{FD}}{\partial x} = \frac{1}{e^{\beta(\epsilon-\mu)} + 1}, \quad (8)$$

where  $x = \beta(\epsilon - \mu)$ ,

$\epsilon$  is the energy,

$\mu$  is the chemical potential,

$\beta = 1/kT$ ,  $T$  is the temperature,

and  $k$  is Boltzmann constant.

$Z$  is the Grand Partition Function

$\bar{n}_{BE}, \bar{n}_{FD}$  are the distributions for number of particles over different quantum energy states at  $T$  for systems of bosons and fermions.

## 3B. Quantum Statistics

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1. Fermi-Dirac

2. Bose-Einstein

- In this work we will present applications of the  $W$  function to free ideal Bose gases.
- We represent the number of particles  $N$ , entropy  $S$ , pressure  $P$ , chemical potential  $\mu$ , and energy  $U$  using the  $Z$  grand partition function:

$$N = kT \frac{\partial(\ln \mathcal{Z})}{\partial \mu}, \quad S = -k \frac{\partial(T \ln \mathcal{Z})}{\partial T}, \quad P = kT \frac{\partial(\ln \mathcal{Z})}{\partial V}$$
$$\mu = -kT \frac{\partial(\ln \mathcal{Z})}{\partial N}, \quad U = kT^2 \frac{\partial(\ln \mathcal{Z})}{\partial T}.$$

- The thermodynamic quantity that gives most insight into the nature of BEC is the fugacity

$$z = e^{\beta \mu}$$

- In the limit where all bosons are in the ground state, the fugacity approaches unity.

# 4. Two, Three and Four Dimensional Ideal Bose Gases

- The reduced particle density for a Bose gas in  $D$  dimensions is given in terms of a polylogarithm of order related to the dimension. The result is:

$$\rho\lambda^D = Li_{m+1}(z) , \quad (9)$$

- Where  $Li_{m+1}(z)$  is the polylogarithm of order  $m + 1$ ,  $m = D/2 - 1$  and  $D$  is the dimension,  $z = e^{\beta\mu}$  is the fugacity of the system,  $\lambda = \sqrt{2\pi\hbar^2/kTM}$  is the de-Broglie wavelength,  $M$  is the mass of the constituent particles, and the quantity  $\rho\lambda^D$  is referred to as the reduced particle density.

The analysis presented here relies on an expansion of the polylogarithms about  $z = 1$ .

If  $m=0,1,2, (D=2,4,6,)$ , as  $z \rightarrow 1$ ,

$$Li_{m+1} = \frac{(-1)^{m+1}}{\Gamma(m+1)} \left( \ln \frac{1}{z} \right)^m \left( \ln \ln \left( \frac{1}{z} \right) - \psi(m+1) + \psi(1) \right) + \sum_{r=0(r \neq m)}^{\infty} \frac{\zeta(m+1-r)(\ln z)^r}{r!},$$

(10)

Where  $\zeta(s)$  is the Riemann Zeta Function,

$\Gamma(s)$  is the Gamma Function, and

$\psi(s)$  (also referred to as the Digamma Function) is the logarithmic derivative of the Gamma Function.

# 4A. Chemical Potential

- In  $D = 2$  and  $D = 4$  dimensions, from  $\rho\lambda^D = Li_{m+1}(z)$ , the reduced particle densities are

$$\rho\lambda^2 = Li_1(z), \quad D = 2, \quad (11a)$$

and

$$\rho\lambda^4 = Li_2(z), \quad D = 4 \quad (11b)$$

Two dimensional case:

$m = 0 \longrightarrow$  the logarithmic derivatives cancel.

as  $r$  increases the summation become small in the following:

$$Li_{m+1} = \frac{(-1)^{m+1}}{\Gamma(m+1)} \left( \ln \frac{1}{z} \right)^m \left( \ln \ln \left( \frac{1}{z} \right) - \psi(m+1) + \psi(1) \right) + \sum_{r=0(r \neq m)}^{\infty} \frac{\zeta(m+1-r)(\ln z)^r}{r!},$$

From the previous eqn. and  $\rho\lambda^4 = Li_2(z)$ ,  $D = 4$  the reduced particle density near  $z = 1$  is

$$\rho\lambda^2 = Li_1(z) \approx -\ln \ln(1/z) + \zeta(0) \ln z \quad D = 2 \quad (12)$$

which can be rewritten as

$$x - 2 \ln x - 2\rho\lambda^2 = 0, \quad x = -\ln z$$

using the fact that  $\zeta(0) = -1/2$

Hence this (12) is of the form (3) with  $n = 1$ ,  $B = -2$  and  $C = -2\rho\lambda^2$  so that its solution is given by (4),

$$x = -\ln z = -2 W_j \left[ -\frac{1}{2} \exp \left( - \left( \frac{-2\rho\lambda^2}{-2} \right) \right) \right]^{1/1} \quad (13)$$

and by the definition of  $z$  we have  $\mu = -kTx$ , then,

$$\mu = 2kTW_j \left( -\frac{1}{2} e^{-\rho\lambda^2} \right), \quad \lambda = \sqrt{2\pi\hbar^2/kTM}, \quad D = 2.$$

$y$  is the argument of the  $W$  function

$$y \rightarrow 0 \Rightarrow W_0(y) \rightarrow 0 \text{ and } W_{-1}(y) \rightarrow -\infty.$$

if  $\mu$  is given in terms of  $W_0$  then

$$T \rightarrow 0 \Rightarrow \lambda \rightarrow \infty \Rightarrow \mu \rightarrow 0$$

$$W_0(0) = 0,$$

Whereas if  $\mu$  is expressed in terms of  $W_{-1}(z)$ , then  $\beta\mu \rightarrow -\infty$

Since we are interested in region  $z \approx 1$  we only consider the solution in terms of  $W_0$ .

$$\mu = -kT \sum_{n=1}^{\infty} \left(\frac{n}{2}\right)^{n-1} \frac{e^{-n\rho\lambda^2}}{n!}. \quad (14)$$

Based on this expression we have the following results in 2-D:

- For real  $\mu$ , treating  $\rho\lambda^2$  as an independent variable, the series (14) has a radius of convergence  $R = 1 - \ln 2$ . Within the circle of convergence,  $|\mu| < 2kT$ . Also,  $\mu$  has a branch point at  $\rho\lambda^2 = 1 - \ln 2$  and a branch cut along the negative real axis:

$$\rho\lambda^2 \in (-\infty, R).$$

- For real values of  $\mu$ , condensation occurs when  $e^{-\rho\lambda^2} \rightarrow 0$ , as expected in two dimensions. This is in agreement  $T \rightarrow 0$ , with the result obtained by Lee.

# Return to chemical potential in 4-D

$$D = 4, m = 1,$$

$$\rho\lambda^4 = \ln\left(\frac{1}{z}\right) \left( \ln \ln\left(\frac{1}{z}\right) - \psi(2) + \psi(1) \right) + \zeta(2), \quad (15)$$

Where  $\psi(2) = \psi(1) + 1$ ,  $\psi(1) = -C$ ,  $C$  is Euler's constant.

Equivalently,

$$x \ln(x) - x + (\zeta(2) - \rho\lambda^4) = 0 \quad x = -\ln z, \quad (16)$$

with  $n = 1$ ,  $B = -1$  and  $C = (\zeta(2) - \rho\lambda^4)$ .

- Therefore the solution to [\(15\)](#) using  $\zeta(2) = \pi^2/6$ .

$$x = -\ln z = \left[ -\frac{1}{(1)(\zeta(2) - \rho\lambda^4)} W_j \left( -(\zeta(2) - \rho\lambda^4)(1) \exp((1)(-1)) \right) \right]^{-1/1},$$

$$= \left( \rho\lambda^4 - \frac{\pi^2}{6} \right) \left[ W_j \left( \left[ \rho\lambda^4 - \frac{\pi^2}{6} \right] e^{-1} \right) \right]^{-1}$$

By the definition of  $W(z)$ ,  $W(z)^{-1} = \exp(W(z))/z$ , so:

$$\mu = -kTx = -kT \exp \left( W_j \left( \frac{1}{e} \left[ \rho\lambda^4 - \frac{\pi^2}{6} \right] \right) + 1 \right) \quad \text{(17)}$$

In general, [\(17\)](#) allows for the possibility of a complex chemical potential. By choosing  $j = \mathbf{0}$  or  $j = \mathbf{-1}$  the chemical potential is real.

For real branches of  $W$ , a solution exists if:

$$W_j [e^{-1} (\rho\lambda^4 - (\pi^2/6))] \rightarrow -\infty,$$

which is only possible if  $j = -1$  and  $e^{-1} (\rho\lambda^4 - \frac{\pi^2}{6}) \rightarrow 0$ .

Recalling that:  $\lambda = \sqrt{2\pi\hbar^2/(MkT)},$

We have:

$$T_c = \sqrt{\frac{24\hbar^4\rho}{k^2M^2}} = \frac{2\hbar^2\sqrt{6\rho}}{kM}. \quad (18)$$

By definition,

$$W_{-1}(z) \in \mathbb{R} \iff z \in [-1/e, 0).$$

This constraint can be expressed in terms of  $T_c$  by noting that in our case it is equivalent to

$$T \in [(\pi/\sqrt{6 + \pi^2})T_c, T_c) \approx [0.789T_c, T_c).$$

For real values,  $T$  must be within about 80% and 100%  $T_c$

## 4B. Pressure

**Pressure  $P$**  is a function of the chemical potential.

$$\frac{\rho}{kT} = \frac{\partial P}{\partial \ln z}$$

As a **polylogarithm**, order of pressure  $P$  depends on the dimension.  
Hence, for two-dimensional case:

$$\frac{P}{kT} = \frac{2}{\lambda^2} Li_2(z) \quad D = 2$$

Note: Volume and Temperature are held fixed

$\rho$  is the number density

$z$  is the fugacity

$N$  is the number of particles

# Two-Dimensional Case of Ideal Bose Gas

The pressure around  $z = 1$  is expressed as:

$$\frac{P}{kT} \approx \frac{2}{\lambda^2} \left[ \ln \left( \frac{1}{z} \right) \left( \ln \left( \ln \left( \frac{1}{z} \right) \right) - 1 \right) + \frac{\pi^2}{6} \right] .$$

The fact that  $\ln z = (1/kT)\mu$ , leads to:

$$\frac{P}{kT} \approx \frac{2}{\lambda^2} \left[ -2W_j \left( -\frac{1}{2} e^{-\rho\lambda^2} \right) \left( \ln \left( -2W_j \left( -\frac{1}{2} e^{-\rho\lambda^2} \right) \right) - 1 \right) + \frac{\pi^2}{6} \right]$$

$$\ln(-2W_j(y)) + W_j(y) = \ln 2 + \ln(-y) .$$

Thus, pressure of a two-dimensional gas is:

$$\frac{P}{kT} \approx \frac{4}{\lambda^2} \left[ W_j \left( -\frac{1}{2} e^{-\rho\lambda^2} \right) \right]^2 + \frac{4(\rho\lambda^2 + 1)}{\lambda^2} W_j \left( -\frac{1}{2} e^{-\rho\lambda^2} \right) + \frac{\pi^2}{3\lambda^2} , \quad D = 2.$$

When  $W_0$  is very small,

$$\frac{P}{kT} = \frac{4(\rho\lambda^2 + 1)}{\lambda^2} W_0 \left( -\frac{1}{2} e^{-\rho\lambda^2} \right) + \frac{\pi^2}{3\lambda^2} , \quad D = 2$$

Note: As limit  $T \rightarrow 0$  ( $\lambda \rightarrow \infty$ ), right hand side  $\rightarrow 0$

- Using series expansion for  $W_0$  about  $z = 0$ , series expansion for pressure about  $\rho\lambda^2 \gg 1$  is:

$$\frac{P}{kT} = -\frac{2(\rho\lambda^2 + 1)}{\lambda^2} \sum_{n=1}^{\infty} \left( \frac{n}{2} \right)^{n-1} \frac{e^{-n\rho\lambda^2}}{n!} + \frac{\pi^2}{3\lambda^2} , \quad D = 2$$

# Four-Dimensional Case of Ideal Bose Gas

Pressure is expressed as:  $\frac{P}{kT} = \frac{2}{\lambda^3} Li_3(z) \quad D = 4$

As mentioned above,

$$Li_{m+1} = \frac{(-1)^{m+1}}{\Gamma(m+1)} \left( \ln \frac{1}{z} \right)^m \left( \ln \ln \left( \frac{1}{z} \right) - \psi(m+1) + \psi(1) \right) + \sum_{r=0(r \neq m)}^{\infty} \frac{\zeta(m+1-r)(\ln z)^r}{r!}$$

pressure is related to fugacity  $\mathbf{z}$  by:

$$\frac{P}{kT} = \frac{2}{\lambda^3} \left[ -\frac{1}{\Gamma(3)} \left( \ln \left( \frac{1}{z} \right) \right)^2 \left( \ln \ln \left( \frac{1}{z} \right) - \psi(3) + \psi(1) \right) + \zeta(3) + \zeta(2) \ln z \right]$$

and by neglecting  $\ln z$  terms of order  $\geq 2$  :

$$\frac{P}{kT} \approx \frac{2\zeta(3)}{\lambda^3} + \frac{2\zeta(2)}{\lambda^3} \ln z$$

Using solution

$$\mu = -kTx = -kT \exp \left( W_j \left( \frac{1}{e} \left[ \rho\lambda^4 - \frac{\pi^2}{6} \right] \right) + 1 \right)$$

equation of state for free ideal Bose gas near  $z=1$  and  $D=4$  is:

$$\boxed{\frac{P}{kT} = \frac{2\zeta(3)}{\lambda^3} - \frac{\pi^2}{3\lambda^3} \exp \left( W_{-1} \left( \frac{1}{e} \left[ \frac{\pi^2}{6} - \rho\lambda^4 \right] \right) + 1 \right)}$$

# Conclusions

- **New representations for chemical potential  $\mu$ ,  $P$ ,  $T$  of Bose Gas**
  - Relationship between chemical potential and  $T$  in terms of  $W$  function in two, three, four dimensions
  - Branch cuts of chemical potential in two dimensions
  - Chemical Potential and Decay in a BEC
  - Series Expansion in terms of quantity  $\exp(-\rho\lambda^2)$  with  $D=2$
  - Expression for small  $T$  below condensation  $T$ 
    - Real Values for chemical potential for  $T \in [0.789T_c, T_c)$
  - Equation of state in BEC regime

- **W function and analysis of BEC of trapped ideal Bose gas**
  - Condensate  $T$  with  $D=1$ , expressed in terms of  $N$ .
  - Chemical Potential with  $D=2$ , express in terms of  $T$  and  $N$ .
- **High  $T$  expansions for  $P$  and  $\rho$  of hard-core bosons,  $D=1$** 
  - Radius of Convergence, Coefficients of Mayer expansions differ from Tonk gas
  - Temperature, chemical potential, pressure in low temperature analysis

- **Applications of Type 1 and Type 2 equations to Bose-Einstein and Maxwell-Boltzmann systems at high temperatures.**
  - Convergence of different statistics in classical unit
  - Complex chemical potential and temperature
  - Incomplete complex analysis for ideal Bose at  $D=2$  &  $D=4$  near condensation
  - Further study on branch cuts, singularities of the thermodynamic functions.
- **Applications of  $W$  function**
- **Obtaining several special solutions of  $W$  function in classical and non-classical limits**
- **Bose and Lambert are no more but Bosons and The Lambert  $W$  will live for ever.**