Toeplitz CAR flows

R. SRINIVASAN
(Joint work with MASAKI IZUMI)
Chennai Mathematical Institute
vasanth@cmi.ac.in

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R. T. Powers, *A nonspatial continuous semigroup of $*-$endomorphisms of B(H).* Publ. Res. Inst. Math. Sci. 23 (1987), 1053-1069.

In 1987 R. T. Powers discovered the first example of a type III E_0-semigroup. Although his purpose is to construct a single type III example, his construction is rather general, and it could produce several E_0-semigroups, by varying the associated quasi-free states. But the problem is to find invariants to distinguish them up to cocycle conjugacy.

B. Tsirelson, *Non-isomorphic product systems.* Advances in Quantum Dynamics (South Hadley, MA, 2002), 273328, Contemp. Math., 335, Amer. Math. Soc., Providence, RI, 2003.

In 2001, Boris Tsirelson constructed a one-parameter family of nonisomorphic product systems of type III. Using previous results of Arveson, this leads to the existence of uncountably many E_0-semigroups of type III, which are mutually non cocycle conjugate. Since then there has been some activity along this direction.

B. V. Rajarama Bhat and R. Srinivasan, *On product systems arising from sum systems,* Infinite dimensional analysis and related topics, Vol. 8, Number 1, March 2005.

M. Izumi, *A perturbation problem for the shift semigroup*. J. Funct. Anal. 251, (2007), 498545.

M. Izumi and R. Srinivasan, *Generalized CCR flows.* Commun. Math. Phys. 281, (2008), 529571.

Here we turn our attention to the first example of a type III E_0-semigroup produced by Powers.

M. Izumi and R. Srinivasan, *Toeplitz CAR flows and type I factorizations*, Kyoto J. Math, Volume 50, Number 1(2010), 1-32.

Toeplitz CAR flows are a class of E_0-semigroups including the first type III example constructed by R. T. Powers. We show that the Toeplitz CAR flows contain uncountably many mutually non cocycle conjugate E_0-semigroups of type III. We also generalize the type III criterion for Toeplitz CAR flows employed by Powers (and later refined by W. Arveson), and consequently show that Toeplitz CAR flows are always either of type I or type III.

H - separable Hilbert space. $B(H)$ - $*-$algebra of all bounded linear operators on H.

An E_0-semigroup on $B(H)$ is a semigroup of 'normal' unital $*-$endomorphisms on $B(H)$, which are weakly continuous.

Definition

$\{\alpha_t : t \geq 0\}$, a family of linear operators on $B(H)$, is an E_0-semigroup if

(0) $\alpha_s \alpha_t = \alpha_{s+t} \;\; \forall\; s, t \in (0, \infty), \quad \alpha_0 = id.$

(i) $\alpha_t(XY) = \alpha_t(X)\alpha_t(Y), \;\; \forall\; X, Y \in B(H), \; t \in (0, \infty).$

(ii) $\alpha_t(X^*) = \alpha_t(X)^*, \;\; \forall\; X \in B(H), \; t \in (0, \infty).$

(iii) $\alpha_t(1) = 1, \;\; \forall\; t \in (0, \infty).$

(iv) For every $t \in (0, \infty)$, α_t is $\sigma-$weakly continuous.

(v) The map $t \mapsto \langle \alpha_t(X)\xi, \eta \rangle$ is continuous as a complex valued function, for every fixed $X \in B(H), \;\; \xi, \eta \in H$.

(Wigner) Every one parameter group of automorphisms $\{\alpha_t : t \in \mathbb{R}\}$ are given by a strongly continuous one parameter unitary group $\{U_t : t \in \mathbb{R}\} \subseteq B(H)$ by

$$\alpha_t(X) = U_t X U_t^*.$$

An analogous statement of Wigner's theorem for an E_0-semigroup would be that the semigroup is completely determined by the set of all intertwining semigroup of isometries called units, which are defined as follows.

Definition

A unit for an E_0-semigroup $\{\alpha_t\}$ acting on $B(H)$ is a strongly continuous semigroup of isometries $\{U_t\}$, which intertwines α and the identity, that is

$$\alpha_t(X)U_t = U_t X, \qquad \forall\ X \in B(H),\ t \geq 0.$$

A subclass of E_0-semigroups, where this analogy is indeed true, are called type I E_0-semigroups.

Due to the existence of type II and type III E_0-semigroups in abundance, it is well known by now that such an analogy does not hold for E_0-semigroups in general.

An E_0-semigroup is called as
-type I if units exist and completely determines the E_0-semigroup (in other words 'generates the product system' associated with the E_0-semigroup, which is a complete invariant).

-type II if units exist but does not completely describe the E_0-semigroup (that is it does not generate the product system).

- type III if there does not exist any units for the E_0-semigroup.

An E_0-semigroup is called as spatial if it admits units.

Let K be a complex Hilbert space.

$\mathcal{A}(K)$ is the CAR algebra over K. $\mathcal{A}(K)$ is the canonical C^*-algebra generated by $\{a(x) : x \in K\}$, determined by the linear map $x \mapsto a(x)$ satisfying relations

$$\begin{aligned} a(x)a(y) + a(y)a(x) &= 0, \\ a(x)a(y)^* + a(y)^*a(x) &= \langle x, y\rangle 1, \end{aligned} \tag{0.1}$$

for all $x, y \in K$.

A quasi-free state ω_A on $\mathcal{A}(K)$, associated with a positive contraction $A \in B(K)$, is the state uniquely determined by the values on the $(n, m)-$point functions, specified as

$$\omega_A(a(x_n)\cdots a(x_1)a(y_1)^*\cdots a(y_m)^*) = \delta_{n,m}det(\langle Ax_i, y_j\rangle).$$

Given a positive contraction, it is a fact that such a state always exists and is uniquely determined by the above relation.

Let (H_A, π_A, Ω_A) be the GNS triple associated with a quasi-free state ω_A on $\mathcal{A}(K)$, and set $\mathcal{M}_A := \pi_A(\mathcal{A}(K))''$.

Fact (i): $\mathcal{M}_A$ is always a factor.

Fact (ii): $\mathcal{M}_A$ is a type I factor if and only if

$$Tr(A - A^2) < \infty.$$

Proposition

(Arveson) Let U_t *be a* C_0-*semigroup of isometries acting on* K*, let* $\alpha = \{\alpha_t : t \geq 0\}$ *be the semigroup of endomorphisms of* $\mathcal{A}(K)$ *defined by*

$$\alpha_t(a(x)) = a(U_t x) \qquad \forall \;\; x \in K, \;\; t \geq 0,$$

and let A *be a positive contraction in* $B(K)$ *satisfying*

(i) $U_t^* A U_t = A.$

(ii) $Tr(A - A^2) < \infty$

Then there is a unique E_0-*semigroup* $\tilde{\alpha} = \{\tilde{\alpha}_t : t \geq 0\}$ *on the type I factor* $\mathcal{M}_A$*, satisfying*

$$\tilde{\alpha}_t(\pi_A(a(x))) = \pi_A(\alpha_t(a(x))), \qquad \forall \;\; t \geq 0, \;\; x \in K.$$

From here onwards $K = L^2((0,\infty), \mathbb{C}^N)$, and by $\{S_t\}$ the C_0-semigroup of isometries of the unilateral shift on K, defined for $f \in K$

$$\begin{aligned} (S_t f)(s) &= 0, \quad s < t, \\ &= f(s-t), \quad s \geq t. \end{aligned}$$

We regard K as a closed subspace of $\tilde{K} = L^2(\mathbb{R}, \mathbb{C}^N)$, and we denote by P_+ the projection from $\tilde{K}$ onto K. We often identify $B(K)$ with $P_+ B(\tilde{K}) P_+$.

For $\Phi \in L^\infty(\mathbb{R}) \otimes M_N(\mathbb{C})$, we define $C_\Phi \in B(\tilde{K})$ by

$$(\widehat{C_\Phi f})(p) = \Phi(p)\hat{f}(p).$$

Then the Toeplitz operator $T_\Phi \in B(K)$ and the Hankel operator $H_\Phi \in B(K, K^\perp)$ with the symbol Φ are defined by

$$T_\Phi f = P_+ C_\Phi f, \quad f \in K,$$

$$H_\Phi f = (1_{\tilde{K}} - P_+) C_\Phi f, \quad f \in K.$$

Theorem (Arveson)

Let $K = L^2((0,\infty), \mathbb{C}^N)$. *A positive contraction* $A \in B(K)$ *satisfies* $tr(A - A^2) < \infty$ *and* $S_t^* A S_t = A$ *if and only if there exists a projection* $\Phi \in L^\infty(\mathbb{R}) \otimes M_N(\mathbb{C})$ *satisfying the following two conditions:*

(i) $A = T_\Phi$,

(ii) *the Hankel operator* H_Φ *is Hilbert-Schmidt.*

We call a symbol Φ satisfying the conditions of the above Theorem as *admissible*.

Theorem

(Essentially Tsirelson)

Let $\Phi \in L^\infty(\mathbb{R}) \otimes M_N(\mathbb{C})$ *be a projection.*

If Φ *is an even differentiable function satisfying*

$$\int_0^\infty Tr(|\Phi'(p)|^2) p dp < \infty,$$

then Φ *is admissible.*

Theorem (Arveson–Powers)

Let $\Phi \in L^\infty(\mathbb{R}) \otimes M_N(\mathbb{C})$ *be an admissible symbol having the limit*

$$\Phi(\infty) := \lim_{|p|\to\infty} \Phi(p).$$

If the Toeplitz CAR flow α^Φ *is spatial, then*

$$\int_{\mathbb{R}} Tr(|\Phi(p) - \Phi(\infty)|^2) dp < \infty.$$

Theorem

Let $\Phi, \Psi \in L^\infty(\mathbb{R}) \otimes M_N(\mathbb{C})$ *be admissible symbols. If*

$$\int_{\mathbb{R}} Tr(|\Phi(p) - \Psi(p)|^2) dp < \infty,$$

then α^Φ *and* α^Ψ *are cocycle conjugate.*

Theorem

Let $\Phi \in L^\infty(\mathbb{R}) \otimes M_N(\mathbb{C})$ *be an admissible symbol. Then the following conditions are equivalent:*

(i) *The Toeplitz CAR flow* α^Φ *is of type* I_N.

(ii) *The Toeplitz CAR flow* α^Φ *is spatial.*

(iii) *There exists a projection* $Q \in M_N(\mathbb{C})$ *satisfying*

$$\int_{\mathbb{R}} Tr(|\Phi(p) - Q|^2)dp < \infty.$$

In particular, every Toeplitz CAR flow is either of type I or type III.

Example

Powers' first example of a type III E_0-semigroup is the Toeplitz CAR flow associated with the symbol

$$\Phi(p) = \frac{1}{2}\begin{pmatrix} 1 & e^{i\theta(p)} \\ e^{-i\theta(p)} & 1 \end{pmatrix},$$

where $\theta(p) = (1+p^2)^{-1/5}$. More generally, if $\theta(p)$ is a real differentiable function satisfying $\theta(-p) = \theta(p)$ for $\forall p \in \mathbb{R}$ and

$$\int_0^\infty |\theta'(p)|^2 p dp < \infty,$$

then the symbol Φ as above is admissible.
For $0 < \nu \leq 1/4$, the symbols Φ_ν, given by $\theta_\nu(p) = (1+p^2)^{-\nu}$ in place of $\theta(p)$ above, give rise to mutually non cocycle conjugate type III E_0-semigroups.

Example

Let $\theta(p)$ be a real smooth function satisfying $\theta(-p) = \theta(p)$ for all $p \in \mathbb{R}$ and $\theta(p) = \log^\alpha |p|$ with $0 < \alpha < 1/2$ for large $|p|$. Then Φ associated with θ in the above example is an admissible symbol without having limit at infinity. While the theorem of Arveson-Powers does not apply to such Φ, now we know that the Toeplitz CAR flow α^Φ is of type III.

$\{\alpha_t\}$ be E_0-semigroup on $B(H)$.
For $t \geq 0$ and for a finite interval $I = (s,t)$, define

$$\mathcal{A}((0,t)) = \alpha_t(B(H))' \cap B(H); \qquad \mathcal{A}(I) = \alpha_s(\mathcal{A}((0,t-s))).$$

The family of local algebras, indexed by bounded open intervals in $(0,a)$, for $a > 0$, forms an invariant for the E_0-semigroup.

Definition

Let H be a Hilbert space. We say that a family of type I subfactors $\{\mathcal{M}_n\}_{n\in\mathbb{N}}$ of $B(H)$ is a *type I factorization* of $B(H)$ if

(i) $\mathcal{M}_n \subset \mathcal{M}'_m$ for any $n, m \in \mathbb{N}$ with $n \neq m$,

(ii) $B(H) = \bigvee_{n\in\mathbb{N}} \mathcal{M}_n$.

We say that a type I factorization $\{\mathcal{M}_n\}_{n\in\mathbb{N}}$ is a *complete atomic Boolean algebra of type I factors* (abbreviated as *CABATIF*) if for any subset $\Gamma \subset \mathbb{N}$, the von Neumann algebra $\bigvee_{k\in\Gamma} \mathcal{M}_k$ is a type I factor.

Example

Let $\{a_n\}_{n=0}^{\infty}$ be a strictly increasing sequence of non-negative numbers starting from 0 and converging to $a < \infty$. Then the family of local algebras $\{\mathcal{A}(a_n, a_{n+1})\}_{n=0}^{\infty}$ is a type I factorization. For a fixed sequence as above, the unitary equivalence class of the type I factorization $\{\mathcal{A}(a_n, a_{n+1})\}_{n=0}^{\infty}$ is an isomorphism invariant of the product system E. In particular, whether it is a CABATIF or not will be used to distinguish concrete type III examples.

Theorem

Let $\Phi \in L^\infty(\mathbb{R}) \otimes M_N(\mathbb{C})$ *be an admissible symbol, and let* $\{a_n\}_{n=0}^\infty$ *be a strictly increasing sequence of non-negative numbers such that* $a_0 = 0$ *and it converges to a finite number* a*. Let* $I_n = (a_n, a_{n+1})$ *and* $O = \bigcup_{n=0}^\infty I_{2n}$*.*

(i) *If*

$$\sum_{n=0}^{\infty} \|(1_{\tilde{K}} - P_{I_n})C_\Phi P_{I_n}\|_2^2 < \infty,$$

then $\{\mathcal{A}_a^\Phi(I_n)\}_{n=1}^\infty$ *is a CABATIF.*

(ii) *If* $\{\mathcal{A}_a^\Phi(I_n)\}_{n=0}^\infty$ *is a CABATIF, then* $\|(1_{\tilde{K}} - P_O)C_\Phi P_O\|_2^2 < \infty$*.*

Theorem

Let $\Phi \in L^\infty(\mathbb{R}) \otimes M_N(\mathbb{C})$ *be an admissible symbol satisfying* $\Phi(p) = \Phi(-p)$ *for all* $p \in \mathbb{R}$, *and let* $0 < \mu < 1$. *We set* $a_0 = 0$, $a_n = \sum_{k=1}^{n} \frac{1}{k^{1/(1-\mu)}}, \quad n \in \mathbb{N}$, *and* $a = \lim_{n\to\infty} a_n$.

(i) *If* $\{\mathcal{A}_a^\Phi(a_n, a_{n+1})\}_{n=0}^\infty$ *is a CABATIF, then*

$$\int_0^\infty Tr(|\Phi(2p) - \Phi(p)|^2)\frac{dp}{p^\mu} < \infty.$$

(ii) *If* Φ *is differentiable and*

$$\int_0^\infty Tr(|\Phi'(p)|^2)p^{2-\mu}dp < \infty,$$

then $\{\mathcal{A}_a^\Phi(a_n, a_{n+1})\}_{n=0}^\infty$ *is a CABATIF.*

Theorem

For $\nu > 0$*, let* $\theta_\nu(p) = (1+p^2)^{-\nu}$*, and let*

$$\Phi_\nu(p) = \frac{1}{2}\begin{pmatrix} 1 & e^{i\theta_\nu(p)} \\ e^{-i\theta_\nu(p)} & 1 \end{pmatrix}.$$

Then Φ_ν *is admissible. Let* $\alpha^\nu := \alpha^{\Phi_\nu}$ *be the corresponding Toeplitz CAR flow.*

(i) *If* $\nu > 1/4$*, then* α^ν *is of type* I_2*.*

(ii) *If* $0 < \nu \leq 1/4$*, then* α^ν *is of type III.*

(iii) *If* $0 < \nu_1 < \nu_2 \leq 1/4$*, then* α^{ν_1} *and* α^{ν_2} *are not cocycle conjugate.*