SOME APPLICATIONS OF HEISENBERG-WEYL OPERATOR CALCULUS AND ORTHOFERMIONS

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HEISENBERG-WEYL OPERATOR CALCULUS APPROACH TO SOLVING DIFFERENTIAL SYSTEMS

EXAMPLE

$$f(w) = w^3/3 - \alpha w^2 + w = z$$

Invert:

$$w = g(z)$$

LAGRANGE-BÜRMANN INVERSION THEOREM

$$f(w) = z$$

f analytic at a point a and $f^{'}(a) \neq 0$.
Then it is possible to invert or solve the equation for w:

$$w = g(z)$$

where g is analytic at $b = f(a)$ and

$$g(z) = a + \sum_{n=1}^{\infty} (\frac{d}{dw})^{n-1} (\frac{w-a}{f(w)-b})^{n} \mid_{u=a} \frac{(z-b)^n}{n!}$$

AN OPERATOR CALCULUS APPROACH

Acting on polynomials in x, define the operators:

$$D = \frac{d}{dx}$$

and

$$X = \textit{multiplication by } x$$

They satisfy

$$[D, X] = DX - XD = I$$

I, identity operator.
D, X generate the Heisenberg-Weyl algebra (HW).

Fix a neighborhood of 0 in **C**.

Take an analytic function $V(z)$ defined there, normalized to

$$V(0) = 0$$

$$V'(0) = 1$$

Denote

$$W(z) = 1/V'(z)$$

and $U(v)$ the inverse function, i.e.,

$$V(U(v)) = v$$

$$U(V(z)) = z$$

Then $V(D)$ is defined by power series as an operator on polynomials in x and

$$[V(D), X] = V'(D)$$

so that

$$[V(D), XW(D)] = I$$

In other words,

$$V = V(D)$$

and $Y = XW(D)$ generate a representation of the HW algebra on polynomials in x.
Basis for the representation:

$$y_n(x) = Y^n 1$$

i.e., Y is a **raising operator**

$$V y_n = n\, y_{n-1}$$

V is the corresponding **lowering operator**
The operator of multiplication by x is given by

$$X = YV'(D) = YU'(V)^{-1}$$

which is a **recursion operator** for the system. Consider a variable A with corresponding partial differential operator ∂_A.

Given V as above, let $\tilde{Y}$ be the vector field $\tilde{Y} = W(A)\partial_A$. Then:

$$\tilde{Y}\, e^{Ax} = xW(A)\, e^{Ax} = xW(D)\, e^{Ax}$$

as any operator function of D acts as a multiplication operator on e^{Ax}.

Important property: Y and $\tilde{Y}$ commute

Iterate:

$$\exp(t\tilde{Y})e^{Ax} = \exp(tY)e^{Ax}. \tag{1}$$

Solve:

$$\dot{A} = W(A) \tag{2}$$

with initial condition $A(0) = A$, then for any smooth function f,

$$e^{t\tilde{Y}} f(A) = f(A(t)).$$

Thus

$$\exp(tY)e^{Ax} = e^{xA(t)}.$$

To solve equation (2), multiply both sides by $V'(A)$ and observe that:

$$V'(A)\,\dot{A} = \frac{d}{dt}\,V(A(t)) = 1.$$

Integrate:

$$V(A(t)) = t + V(A) \qquad \text{or} \qquad A(t) = U(t + V(A)).$$

Writing v for t, we have:

$$\exp(vY)e^{Ax} = e^{xU(v+V(A))}. \tag{3}$$

Set $A = 0$:

$$\exp(vY)1 = e^{xU(v)}$$

and

$$e^{vY}1 = \sum_{n=0}^{\infty} \frac{v^n}{n!}\, y_n(x).$$

Expansion of the exponential of the inverse function:

$$e^{xU(v)} = \sum_{n=0}^{\infty} \frac{v^n}{n!} y_n(x)$$

or

$$\sum_{m=0}^{\infty} \frac{x^m}{m!} (U(v))^m = \sum_{n=0}^{\infty} \frac{v^n}{n!} y_n(x). \tag{4}$$

Alternative approach to inversion of the function $V(z)$ rather than using Lagrange's formula.

The coefficient of $x^m/m!$ yields the expansion of $(U(v))^m$. $U(v)$ is given by the coefficient of x on the right-hand side.

Theorem

The coefficient of $x^m/m!$ in $Y^n 1$ is equal to $\tilde{Y}^n A^m\big|_{A=0}$, each giving the coefficient of $v^n/n!$ in the expansion of $U(v)^m$.

The same idea works in several variables.

We have

$$\mathbf{V}(\mathbf{z}) = (V_1(z_1, \ldots, z_N), \ldots, V_N(z_1, \ldots, z_N))$$

analytic in a neighborhood of 0 in $\mathbf{C}^N$.
Jacobian matrix

$$\left(\frac{\partial V_i}{\partial z_j}\right)$$

by V' and its inverse by W.
The variables

$$Y_i = \sum_{k=1}^{N} x_k W_{ki}(D)$$

commute and act as raising operators for generating the basis $y_{\mathbf{n}}(\mathbf{x})$.

$$Y_i y_{\mathbf{n}} = y_{\mathbf{n}+\mathbf{e}_i}$$

And

$$V_i = V_i(\mathbf{D})$$

$$\mathbf{D} = (D_1, \ldots, D_N)$$

are lowering operators:

$$V_i y_{\mathbf{n}} = n_i \, y_{\mathbf{n}-\mathbf{e}_i}$$

Denote $\sum_i a_i b_i$ by $a \cdot b$. With variables A_i and corresponding partials ∂_i, define the vector fields

$$\tilde{Y}_i = \sum_k W_{ki}(A) \partial_k .$$

For a vector field $\tilde{Y} = \sum_i W_i(A) \partial_i$, we have the identities

$$\tilde{Y} \, e^{A \cdot x} = x \cdot W(A) \, e^{A \cdot x} = x \cdot W(D) \, e^{A \cdot x}$$

The method of characteristics applies as in one variable and

$$\exp(v \cdot Y) e^{A \cdot x} = e^{x \cdot U(v + V(A))}.$$

Thus, we have the expansion

$$\exp(x \cdot U(v)) = \sum_{\mathbf{n}} \frac{\mathbf{v}^{\mathbf{n}}}{\mathbf{n}!} y_{\mathbf{n}}(\mathbf{x}). \tag{5}$$

The k^{th} component, U_k, of the inverse function is given by the coefficient of x_k in the above expansion.

Important feature of our approach: to get an expansion to a given order requires knowledge of the expansion of W **just to that order**

This allows for streamlined computations.

For polynomial systems $\mathbf{V}$, V' will have polynomial entries, and W will be rational in $\mathbf{z}$.

Raising operators will be rational functions of $\mathbf{D}$, linear in $\mathbf{x}$.

Thus the coefficients of the expansion of the entries W_{ij} of W are computed by finite-step recurrences.

EXAMPLES

Example 1

Let

$$V = z^3/3 - \alpha z^2 + z$$

Then

$$V' = z^2 - 2\alpha z + 1$$

Thus

$$W = \frac{1}{1 - 2\alpha z + z^2} = \sum_{n=0}^{\infty} z^n U_n(\alpha),$$

where U_n are Chebyshev polynomials of the second kind.

Specializing α provides interesting cases.

For example, let

$$\alpha = \cos(\pi/4)$$

or

$$V = z^3/3 - z^2/\sqrt{2} + z$$

Then the coefficients in the expansion of W are periodic with

period 8 and

$$W = \frac{1 + z^2 + \sqrt{2}\, z}{1 + z^4}$$

The coefficient of x in the polynomials y_n yield the coefficients in the expansion of the inverse U.
Here are some polynomials starting with $y_0 = 1$, $y_1 = x$:

$$\begin{aligned} y_2 &= x^2 + x\sqrt{2}, \quad y_3 = x^3 + 3x^2\sqrt{2} + 4x, \\ y_4 &= x^4 + 6x^3\sqrt{2} + 22x^2 + 10x\sqrt{2}, \\ y_5 &= x^5 + 10x^4\sqrt{2} + 70x^3 + 90x^2\sqrt{2} + 40x, \\ y_6 &= x^6 + 15x^5\sqrt{2} + 170x^4 + 420x^3\sqrt{2} + 700x^2 - 140x\sqrt{2}. \end{aligned}$$

This gives to order 6:

$$U(v) = \left(v + \frac{2}{3}v^3 + \frac{1}{3}v^5 + \ldots\right) + \sqrt{2}\left(\frac{1}{2}v^2 + \frac{5}{12}v^4 - \frac{7}{36}v^6 + \ldots\right.$$

This expansion gives approximate solutions to

$$z^3/3 - z^2/\sqrt{2} + z - v = 0$$

for v near 0.

Eample 2
Inversion of the Chebyshev polynomial

$$T_3(z) = 4z^3 - 3z$$

can be used as the basis for solving general cubic equations. We have, with

$$V(z) = 4z^3 - 3z$$

$$W(z) = \frac{-1}{3}\frac{1}{1-4z^2} = \frac{-1}{3}\sum_{n=0}^{\infty} 4^n z^{2n}$$

So

$$y_1 = (-1/3)x$$

$$y_2 = (1/9)x^2$$

$$y_3 = (-1/27)(x^3 + 8x)$$

, etc. We find

$$U(v) = -\frac{1}{3}v - \frac{4}{81}v^3 - \frac{16}{729}v^5 - \frac{256}{19683}v^7 - \cdots$$

In this case, we can find the expansion analytically.
To solve $T_3(z) = v$, write

$$T_3(\cos\theta) = \cos(3\theta) = v$$

Invert to get, for integer k,

$$\theta = (1/3)(2\pi k \pm \arccos v)$$

with arccos denoting the principal branch.
Then

$$z = \cos((1/3)(2\pi k \pm \arccos v))$$

We want a branch with $v = 0$ corresponding to $z = 0$.

With $\arccos 0 = \pi/2$, we want the argument of the cosine to be $\pi/2 + \pi l$, for some integer l.
This yields the condition

$$\frac{1}{3} = \frac{2l+1}{4k \pm 1}$$

Taking $l = 0$, we get $k = 1$, with the minus sign.
Namely,

$$U(v) = \cos((1/3)(2\pi - \arccos v))$$

Use hypergeometric functions and rewrite, we get:

$$U(v) = -\frac{1}{3}\sum_{n=0}^{\infty} \binom{3n}{n} (\frac{4}{27})^n \frac{v^{2n+1}}{2n+1}.$$

If we generate the polynomials y_n, we find the expansion of $U(v)^m$ to any order.

Eample 3

A similar approach is interesting for Chebyshev polynomial $T_n(z)$.

$$F(v) = \cos(\lambda(\mu \pm \arccos v))$$

satisfies the hypergeometric differential equation

$$(1 - v^2)\, F'' - v\, F' + \lambda^2\, F = 0$$

which can be written in the form

$$[(vD_v)^2 - D_v^2]F = \lambda^2\, F$$

with here D_v denoting d/dv.
For integer λ, this is the differential equation for the corresponding Chebyshev polynomial.
In general, these are **Chebyshev functions**.
For $F(0) = 0$, take $\mu = 2\pi k$, we require

$$\lambda = \frac{2l+1}{4k \pm 1}$$

With $F'(0) = \pm\lambda$, we have the solution

$$F(v) = \pm\lambda v \, {}_2F_1\left(\left. \begin{matrix} \frac{1+\lambda}{2}, \frac{1-\lambda}{2} \\ \frac{3}{2} \end{matrix} \right| v^2 \right).$$

USING MAPLE

For symbolic computation using Maple, one can use the **Ore-Algebra** package.

1. **First fix the degree of approximation. Expand W as a polynomial to that degree.**
2. **Declare the Ore algebra with one variable, x, and one derivative, D.**
3. **Define the operator $xW(D)$ in the algebra.**
4. **Iterate starting with $y_0 = 1$ using the applyopr command.**
5. **Extract the coefficient of $x^m/m!$ to get the expansion of $U(v)^m$.**

ALGORITHM AS A MATRIX COMPUTATION

Fix the order of approximation n.
Cut off the expansion

$$W(z) = w_0 + w_1 z + w_2 w^2 + \cdots + w_k z^k + \cdots$$

at $w_n z^n$.
Let the matrix

$$W = \begin{pmatrix} w_1 & w_0 & 0 & \dots & 0 \\ w_2 & w_1 & w_0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ w_{n-1} & w_{n-2} & w_{n-3} & \dots & w_0 \\ w_n & w_{n-1} & w_{n-2} & \dots & w_1 \end{pmatrix}.$$

Define the auxiliary diagonal matrices

$$P = \begin{pmatrix} 1! & 0 & \dots & 0 \\ 0 & 2! & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & n! \end{pmatrix}, \quad M = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & n \end{pmatrix},$$

$$Q = \begin{pmatrix} 1/\Gamma(1) & 0 & \dots & 0 \\ 0 & 1/\Gamma(2) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1/\Gamma(n) \end{pmatrix}.$$

Note that

$$QP = M$$

Denoting

$$y_k(x) = \sum c_j^{(k)} x^j$$

we have:

$$[c_1^{(k+1)}, c_2^{(k+1)}, \dots, c_n^{(k+1)}] = [c_1^{(k)}, c_2^{(k)}, \dots, c_n^{(k)}]PWQ$$

The condition $U(0) = 0$ gives $y_0 = 1$.

Then $y_1 = XW(D)y_0$ yields $y_1 = w_0 x$.

We see that $c_0^{(k)} = 0$ for $k > 0$.

We iterate as follows:

1. Start with w_0 times the unit vector $[1, 0, \ldots, 0]$ of length n.
2. Multiply by W.
3. Iterate, multiplying on the right by MW at each step.
4. Finally, multiply on the right by Q.

The top row gives the coefficients of the expansion of $U(v)$ to order n.

HIGHER ORDER EXAMPLE

$$\begin{aligned} V_1 &= z_1 + z_2^2/2 \\ V_2 &= z_2 - z_1 z_2 \end{aligned}$$

So

$$V' = \begin{pmatrix} 1 & z_2 \\ -z_2 & 1 - z_1 \end{pmatrix} \quad \text{and} \quad W = \frac{1}{1 - z_1 + z_2^2} \begin{pmatrix} 1 - z_1 & -z_2 \\ z_2 & 1 \end{pmatrix}.$$

$$\begin{aligned} Y_1 &= \left(x_1(1 - D_1)) + x_2 D_2\right)(1 - D_1 + D_2^2)^{-1} \\ Y_2 &= (-x_1 D_2 + x_2)(1 - D_1 + D_2^2)^{-1} \end{aligned}$$

Expanding

$$(1 - D_1 + D_2^2)^{-1} = \sum_{n=0}^{\infty}(D_1 - D_2^2)^n$$

yields, with $y_{00} = 1$,

$$y_{01} = x_2, \qquad y_{10} = x_1,$$
$$y_{02} = x_2^2 - x_1, \quad y_{11} = x_2 + x_1 x_2, \quad y_{20} = x_1^2.$$

Thus

$$\exp\big(\mathbf{x}\cdot\mathbf{U}(\mathbf{v})\big) = 1 + x_1 v_1 + x_2 v_2 \\ + (x_2 + x_1 x_2) v_1 v_2 + (x_2^2 - x_1)\frac{v_1^2}{2} + x_1^2 \frac{v_2^2}{2} + \cdots ,$$

so

$$U_1(\mathbf{v}) = v_1 - v_1^2/2 + \cdots$$
$$U_2(\mathbf{v}) = v_2 + v_1 v_2 + \cdots$$

ORTHOFERMIONS and FINITE-DIMENSIONAL CALCULUS

ROTA
The Umbral Calculus (Advances in Mathematics, 1978).
Finite Operator Calculus, 1975.

TEKIN, AYDIN, and ARIK
J. Physics A, 2007.

Start with a set of operators

$$\{c_1, \ldots, c_p\}$$

p a positive integer.
Form the star-algebra generated by the $\{c_i\}$ modulo the following relations

$$\begin{aligned} c_i c_j &= 0 \\ c_i c_j^* + \delta_{ij} \sum_{k=1}^{p} c_k^* c_k &= \delta_{ij} \mathbf{1} \end{aligned} \tag{6}$$

$\mathbf{1}$: identity operator.
Set

$$\Pi = \mathbf{1} - \sum_{k=1}^{p} c_k^* c_k$$

This last relation writes as:

$$c_i c_j^* = \delta_{ij} \Pi$$

and:

$$\Pi^2 = \Pi$$

i.e., Π is a projection . It follows that

$$\Pi c_k = c_k$$

and

$$c_i c_j^* c_k = \delta_{ij} \, c_k$$

Set

$$\begin{aligned} a &= c_1 + \sum_{k=2}^{p} k \, c_{k-1}^* c_k \\ a^\dagger &= c_1^* + \sum_{k=2}^{p} c_k^* c_{k-1} \end{aligned}$$

$$aa^\dagger - a^\dagger a = \mathbf{1} - (p+1)\, c_p^* c_p$$

and we get:

$$aa^\dagger a - a^\dagger aa = a$$

CALCULUS with MATRICES

Restrict the differentiation operator to the finite-dimensional space of polynomials of degree less than or equal to p.
Use the standard basis $\{1, x, x^2, \ldots, x^p\}$.
For $p = 4$, we have

$$\hat{D} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$DXD - XDD = D$$

The matrix of X for $p = 4$,

$$\hat{X} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Note that

$$\hat{X}^{p+1} = 0$$

To keep in line with the powers of x, label the basis elements starting from 0

e_k: column vector with the only nonzero entry equal to 1 in the $(k+1)^{\mathrm{st}}$ position.

Vacuum state: $\Omega = \mathbf{e}_0$, satisfying $\hat{D}\Omega = 0$.

And

$$\hat{X}^k \Omega = e_k$$

for $1 \le k \le p$

these are raising and lowering operators satisfying

$$\hat{X} e_k = e_{k+1} \theta_{kp}$$

$$\hat{D} e_k = k x e_{k-1}$$

where $\theta_{ij} = 1$ if $i < j$, zero otherwise.
With the inner product

$$\langle e_n, e_m \rangle = \delta_{nm}\, n!$$

we have

$$\hat{D}^* = \hat{X}$$

Let E_{ij}: the standard unit matrices with all but one entry equal to zero, $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$, $1 \le i, j, k, l \le p + 1$.

Connection with orthofermions is given by the $(p+1) \times (p+1)$ matrix realization

$$\hat{c}_i = E_{1\,i+1}$$

for $1 \leq i \leq p$. The orthofermion relations hold and particularly for this realization

$$\hat{c}_i^* \hat{c}_j = E_{i+1\,j+1}$$

Remark:
$\hat{\Pi} = E_{11}$ and the star-algebra generated by the $\hat{c}_i$ is the full matrix algebra.

Theorem

For $p > 0$, let D and X be $(p+1) \times (p+1)$ matrices defined by $D = \sum_{k=1}^{p} k\, E_{k\,k+1}$, $X = \sum_{k=1}^{p} E_{k+1\,k}$. Then the Lie algebra generated by $\{X, D\}$ is $sl(p+1)$.

EXAMPLES

Example 1

The *number operator* is XD.
For $p = 4$:

$$\hat{X}\hat{D} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix}$$

This operator multiplies $\mathbf{e}_n$ by n, for $0 \leq n \leq 4$.
In general:

$$\hat{X}\hat{D} = \sum_{n=0}^{p} n\, E_{n+1\, n+1}$$

which multiplies $\mathbf{e}_n$ by n, for $0 \leq n \leq p$.

Example 2

The Hermite polynomials, occurring in oscillator wave functions, are eigenfunctions of the *Ornstein-Uhlenbeck operator*, $XD - tD^2$, $t > 0$, which for $p = 4$ takes the form

$$\begin{pmatrix} 0 & 0 & -2t & 0 & 0 \\ 0 & 1 & 0 & -6t & 0 \\ 0 & 0 & 2 & 0 & -12t \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix}$$

The eigenvector for each eigenvalue $\lambda = 0, 1, 2, 3, 4$ gives the coefficients of the corresponding polynomial $H_\lambda(x, t)$.
The family of polynomials $\{H_\lambda(x, t)\}_{\lambda \in \mathbb{N}}$ provide an

orthogonal basis for L^2 with respect to the Gaussian measure with mean zero and variance t.

Example 3

The *translation operator* $T_t = e^{tD}$ acts on functions as $e^{tD}f(x) = f(x+t)$. For $p = 4$,

$$\hat{T}_t = \begin{pmatrix} 1 & t & t^2 & t^3 & t^4 \\ 0 & 1 & 2t & 3t^2 & 4t^3 \\ 0 & 0 & 1 & 3t & 6t^2 \\ 0 & 0 & 0 & 1 & 4t \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

generally, with columns given by binomial coefficients times powers of t, corresponding to the action $x \longrightarrow x + t$ on the basis polynomials x^j.

The matrix $\hat{T}_t$ can be computed as the exponential of $t\hat{D}$ defined as a power series:

$$\mathbf{1} + t\hat{D} + t^2\hat{D}^2/2! + \cdots$$

Example 4

The Gegenbauer polynomials satisfy

$$[(XD+\alpha)^2 - D^2]C_n^\alpha(x) = (n+\alpha)^2\, C_n^\alpha(x)$$

Thus we have the *Gegenbauer operator,*
$G_\alpha = (XD+\alpha)^2 - D^2$, which for $p = 4$ takes the form

$$\hat{G}_\alpha = \begin{pmatrix} \alpha^2 & 0 & -2 & 0 & 0 \\ 0 & (1+\alpha)^2 & 0 & -6 & 0 \\ 0 & 0 & (2+\alpha)^2 & 0 & -12 \\ 0 & 0 & 0 & (3+\alpha)^2 & 0 \\ 0 & 0 & 0 & 0 & (4+\alpha)^2 \end{pmatrix}$$

where the spectrum is evident along the diagonal.

Up to order p, one obtains the Gegenbauer polynomials with coefficients given by the eigenvectors of $\hat{G}_\alpha$.

Multivariable calculus with matrices

Extend to N variables.
For matrices, A, B, the tensor product $A \otimes B$ denotes the *Kronecker product* of the two matrices.

If A is $n \times n$, and B is $m \times m$, then $A \otimes B$ is $nm \times nm$ with entries formed by replacing each entry a_{ij} in A with the block matrix $a_{ij}B$. For products of more than two matrices, we conventionally associate to the left.

For a fixed p:

$$(p+1) \times (p+1)$$

matrices $\hat{D}$ and $\hat{X}$I: *the*(p+1)×$(p+1)$ identity matrix.

$$\begin{aligned} \hat{D}_j &= I \otimes I \otimes \cdots \otimes \hat{D} \otimes I \cdots \otimes I \qquad (\hat{D} \text{ in the } j^{\text{th}} \text{ spot}) \\ \hat{X}_j &= I \otimes I \otimes \cdots \otimes \hat{X} \otimes I \cdots \otimes I \qquad (\hat{X} \text{ in the } j^{\text{th}} \text{ spot}) \end{aligned}$$

$\hat{D}_j$ and $\hat{X}_j$ satisfy the orthofermion relations while

$$[\hat{D}_j, \hat{X}_i] = [\hat{X}_j, \hat{X}_i] = [\hat{D}_j, \hat{D}_i] = 0$$

for $i \neq j$

Analytic representations of the HW-algebra. Canonical polynomials

These are infinite-dimensional representations in the sense that they act on a basis for the vector space of polynomials in a given set of variables

$$\{x_1, x_2, \ldots, x_N\}$$

Use of *canonical variables* which are functions of X and D obeying the HW relations on an infinite-dimensional space, which restricts to the orthofermion relation on spaces of polynomials in x of a given bounded degree.

Notation

We use the convention of summing over repeated Greek indices, *irrespective of position.*

Given $V \colon \mathbf{C}^N \longrightarrow \mathbf{C}^N$,
$V(z) = (V_1(z_1, \ldots, z_N), \ldots, V_N(z_1, \ldots, z_N))$ holomorphic in a neighborhood of the origin, satisfying $V(0) = 0$, construct an associated abelian family of dual vector fields.
Corresponding to the operators X_i of multiplication by x_i, we have the partial differentiation operators, D_i.

In this context, a function of $x = (x_1, \ldots, x_N)$, $f(x)$, is identified with $f(X)1$, the operator of multiplication by $f(X)$ acting on the *vacuum state* 1, with $D_i 1 = 0$, for all $1 \leq i \leq N$.
Define operators

$$V(D) = (V_1(D_1, \ldots, D_N), \ldots, V_N(D_1, \ldots, D_N))$$

These are canonical lowering operators, corresponding to differentiation.

Jacobian:

$$\left(\frac{\partial V_i}{\partial z_j}\right)$$

by $V'(z)$, let $W(z) = (V'(z))^{-1}$, be the inverse (matrix inverse) Jacobian.

The boson commutation relations extend to

$$[V_i(D), X_j] = \frac{\partial V_i}{\partial D_j}$$

Define the operators

$$Y_i = X_\mu W_{\mu i}(D)$$

These are canonical raising operators, corresponding to multiplication by X_i.

$$[V_i, Y_j] = \delta_{ij}\mathbf{1}$$

Canonical system of raising and lowering operators:

$$\{Y_j\}$$

$$\{V_i\}$$

$$1 \leq i, j \leq N$$

Essential feature:

$$[Y_i, Y_j] = [V_i, V_j] = 0$$

Remark.

Exchanging D with X is a formal Fourier transformation and turns the variables Y_i into the vector fields $\tilde{Y}_i = W(x)_{\mu i} \frac{\partial}{\partial x_\mu}$.
The Y_i are *dual vector fields* .

Notation

Complement the standard notations used along with V and W, letting U denote the inverse function to V. I.e.,

$$U \circ V = V \circ U = \mathrm{id}$$

Explicitly:

$$U(V(z)) = z$$

Since

$$W = V'^{-1}$$

we have

$$W(z) = U'(V(z))$$

In other words, converting from z to V acting on functions of the canonical variables Y_i, gives the *recurrence relation*

$$X = Y\, U'(V)^{-1}$$

Multi-index notation, $n = (n_1, \ldots, n_N)$,

$$v^n = v_1^{n_1} v_2^{n_2} \cdots v_N^{n_N}$$

Main formula:

$$\exp(v_\mu Y_\mu)\, 1 = \exp x_\mu U_\mu(v) = \sum_{n \geq 0} \frac{v^n}{n!}\, y_n(x)$$

This expansion defines the *canonical polynomials*:
$y_n(x) = Y^n\, 1$.

Canonical Appell systems

An *Appell system*, $\{h_n(x)\}$, in one variable is a system of polynomials providing a basis for the vector space of polynomials with

$$\deg h_n = n$$

$$n = 0, 1, 2 \dots$$

such that

$$Dh_n = nh_{n-1}$$

Defining the raising operator R by

$$Rh_n = h_{n+1}$$

we have

$$[D, R] = \mathbf{1}$$

thus a representation of the HW-algebra.

Introduce a Hamiltonian $H(z)$.

Only requirement: analyticity in a neighborhood of the origin in $\mathbb{C}^N$.

We have the time-evolution:

$$\exp\left(-tH(D)\right) e^{xU(v)} = e^{xU(v)-tH(U(v))} = \sum_{n\geq 0} \frac{v^n}{n!}\, y_n(x,t)$$

An Appell system of polynomials has a generating function of the form

$$\exp\left(xz - tH(z)\right) = \sum_{n\geq 0} \frac{z^n}{n!}\, h_n(x,t)$$

For the canonical Appell system we have

$$\exp\left(xz - tH(z)\right) = \sum_{n\geq 0} \frac{V(z)^n}{n!}\, y_n(x,t)$$

Take

$$z = U(v)$$

which we interpret as changing to canonical variables.

Each of the polynomials $y_n(x,t)$ is a solution of the evolution equation:

$$\frac{\partial u}{\partial t} + H(D)\, u = 0$$

CANONICAL CALCULUS with MATRICES

First consider the case $N = 1$.

$V(z)$ analytic function in a neighborhood of the origin in $\mathbb{C}$, normalized to $V(0) = 0$, $V'(0) \neq 0$.
Let $W(z) = 1/V'(z)$ have the Taylor expansion

$$W(z) = w_0 + w_1 z + \cdots + w_k z^k + \cdots$$

The corresponding canonical variable is $Y = XW(D)$, satisfying

$$[V(D), Y] = \mathbf{1}$$

The canonical basis polynomials are

$$y_n(x) = Y^n 1$$

$n \geq 0$.

Fix the order p.
Let $\hat{W} = W(\hat{D})$.
Employ the algebra generated by the operators

$$\hat{V} = V(\hat{D})$$

and

$$\hat{Y} = \hat{X}\hat{W}$$

Since

$$\hat{D}^{p+1} = 0$$

The operators $\hat{V}$ and $\hat{W}$ are polynomials in $\hat{D}$.
Similarly, since

$$\hat{X}^{p+1} = 0$$

the polynomials $y_n(\hat{X})$ are truncated if $n > p$.
For $n \leq p$, the correspondence between the polynomials $y_n(x)$ and vectors $\hat{y}_n = y_n(\hat{X})\mathbf{e}_0$ is exact.
The vector $\hat{y}_n$ gives the coefficients of the polynomial $y_n(x)$.

Remark.
Up to order p, the operator $\hat{X}$ never acts on a power of x greater than p.

EXAMPLES

Example 1

$$V(z) = e^z - 1\,, \qquad U(v) = \log(1+v)$$

so

$$W(z) = e^{-z}$$

$$Y = Xe^{-D}$$

The relation

$$X = YU'(V)^{-1}$$

reads

$$X = Y + YV$$

or

$$xy_n = y_{n+1} + ny_n$$

yielding the recurrence

$$y_{n+1} = (x - n)y_n$$

for $n > 0$.

From $y_0 = 1$, calculate

$$y_n(x) = x(x-1)\cdots(x-n+1).$$

For $p = 4$, with $\hat{Y} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 & 1 \\ 0 & 1 & -2 & 3 & -4 \\ 0 & 0 & 1 & -3 & 6 \\ 0 & 0 & 0 & 1 & -4 \end{pmatrix}$ we get

$$\hat{Y}^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -4 & 8 & -15 \\ 1 & -3 & 8 & -20 & 43 \\ 0 & 1 & -5 & 18 & -46 \\ 0 & 0 & 1 & -7 & 22 \end{pmatrix}$$

$$\hat{Y}^3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 2 & -6 & 18 & -53 & 126 \\ -3 & 11 & -39 & 130 & -327 \\ 1 & -6 & 29 & -116 & 313 \\ 0 & 1 & -9 & 46 & -134 \end{pmatrix}$$

$$\hat{Y}^4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ -6 & 24 & -95 & 345 & -900 \\ 11 & -50 & 219 & -845 & 2255 \\ -6 & 35 & -180 & 754 & -2070 \\ 1 & -10 & 65 & -300 & 849 \end{pmatrix}$$

and

$$\hat{Y}^5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 24 & -119 & 559 & -2244 & 6074 \\ -50 & 269 & -1333 & 5497 & -15016 \\ 35 & -215 & 1149 & -4907 & 13559 \\ -10 & 75 & -440 & 1954 & -5466 \end{pmatrix}$$

with the first column giving the coefficients of the corresponding polynomial y_n, where, since the leading coefficient equals one, we can see the truncation beginning in this last.

Example 2

Gaussian with drift $\alpha > 0$,

$$V(z) = \alpha z - z^2/2\,, \qquad U(v) = \alpha - \sqrt{\alpha^2 - 2v}$$

the minus sign taken in $U(v)$ to have $U(0) = 0$.
Then

$$W(z) = \frac{1}{\alpha - z}$$

and

$$\hat{Y} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ \alpha^{-1} & \alpha^{-2} & 2\,\alpha^{-3} & 6\,\alpha^{-4} & 24\,\alpha^{-5} \\ 0 & \alpha^{-1} & 2\,\alpha^{-2} & 6\,\alpha^{-3} & 24\,\alpha^{-4} \\ 0 & 0 & \alpha^{-1} & 3\,\alpha^{-2} & 12\,\alpha^{-3} \\ 0 & 0 & 0 & \alpha^{-1} & 4\,\alpha^{-2} \end{pmatrix}$$

Powers of $\hat{Y}$ yield the canonical polynomials, the first few of which are

$$
\begin{aligned}
y_1 &= \frac{x}{\alpha} \\
y_2 &= \frac{x}{\alpha^3} + \frac{x^2}{\alpha^2} \\
y_3 &= 3\,\frac{x}{\alpha^5} + 3\,\frac{x^2}{\alpha^4} + \frac{x^3}{\alpha^3} \\
y_4 &= 15\,\frac{x}{\alpha^7} + 15\,\frac{x^2}{\alpha^6} + 6\,\frac{x^3}{\alpha^5} + \frac{x^4}{\alpha^4} \\
y_5 &= 105\,\frac{x}{\alpha^9} + 105\,\frac{x^2}{\alpha^8} + 45\,\frac{x^3}{\alpha^7} + 10\,\frac{x^4}{\alpha^6} + \frac{x^5}{\alpha^5}
\end{aligned}
$$

These are a scaled variation of Bessel polynomials and:

$$U'(V)^{-1} = \alpha\left(1 - \frac{2V}{\alpha^2}\right)^{1/2}$$

Thus, expanding and rearranging the relation

$$X = YU'(V)^{-1}$$

$$\alpha Y = X + \alpha Y \left(\frac{V}{\alpha^2} + \frac{1}{2}\frac{V^2}{\alpha^4} + \frac{1}{2}\frac{V^3}{\alpha^6} + \frac{5}{8}\frac{V^4}{\alpha^8} + \frac{7}{8}\frac{V^5}{\alpha^{10}} + \frac{21}{16}\frac{V^6}{\alpha^{12}} + \frac{33}{16} \right.$$

which translates to

$$\begin{aligned} \alpha\, y_{n+1} &= xy_n + \frac{n}{\alpha}\, y_n + \frac{n(n-1)}{2\alpha^3}\, y_{n-1} + \frac{n(n-1)(n-2)}{2\alpha^5}\, y_{n-2} + \ldots \\ &= xy_n + \frac{n}{\alpha}\, y_n + \sum_{k=2}^{n} \binom{n}{k} \frac{(2k-3)!}{\alpha^{2k-1}}\, y_{n-k+1} \end{aligned}$$

Example 3

LambertW function, $\mathcal{W}$.
Take

$$V(z) = ze^{-z}$$

Then

$$U(v) = -\mathcal{W}(-v)$$

$$Y = Xe^{D}(I - D)^{-1}$$

and with $p = 7$ the corresponding matrix

$$\hat{Y} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 5 & 16 & 65 & 326 & 1957 & 13700 \\ 0 & 1 & 4 & 15 & 64 & 325 & 1956 & 13699 \\ 0 & 0 & 1 & 6 & 30 & 160 & 975 & 6846 \\ 0 & 0 & 0 & 1 & 8 & 50 & 320 & 2275 \\ 0 & 0 & 0 & 0 & 1 & 10 & 75 & 560 \\ 0 & 0 & 0 & 0 & 0 & 1 & 12 & 105 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 14 \end{pmatrix}$$

One can show that

$$y_n = x(x+n)^{n-1}$$

and that the relation $X = YU'(V)^{-1}$ leads to the recurrence

$$y_{n+1} = (x + 2n)y_n + \sum_{k=1}^{n-1} \binom{n}{k+1} k^k \, y_{n-k} \ .$$

THANKS!