Transfer Functions associated to Markov Chains

Rolf Gohm
Institute of Mathematics and Physics
Aberystwyth University

International Conference on Quantum Probability and Related Topics
August 14-17, 2010
JNCASR, Bangalore

Plan

In this talk we want to explore some connections between

Markov processes in quantum probability

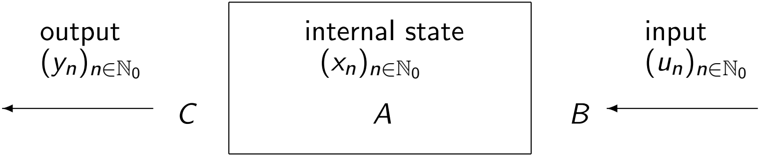
multivariate operator theory

concepts from **control** theory

We do this by examining a rather concrete toy model and we focus on the notion of a **transfer function**.

Linear Systems

$$
\begin{aligned}
x_{n+1} &= A\,x_n + B\,u_n \\
y_n &= C\,x_n + D\,u_n
\end{aligned}
$$



Given x_0 and $(u_n)_{n\in\mathbb{N}_0}$ we can use these equations to compute $(x_n)_{n\in\mathbb{N}_0}$ and $(y_n)_{n\in\mathbb{N}_0}$ recursively.

Transfer Functions

Well known technique in system theory: the z-**transform**. Replace a sequence $(x_n)_{n\in\mathbb{N}_0}$ by a function

$$\sum_{n=0}^{\infty} x_n\, z^n \quad =: \hat{x}(z)$$

Transfer Functions

Well known technique in system theory: the z-**transform**. Replace a sequence $(x_n)_{n\in\mathbb{N}_0}$ by a function

$$\sum_{n=0}^{\infty} x_n\, z^n \quad =: \hat{x}(z)$$

Then if $x(0) = 0$

$$\begin{aligned} z^{-1}\,\hat{x}(z) &= A\,\hat{x}(z) + B\,\hat{u}(z) \\ \hat{y}(z) &= C\,\hat{x}(z) + D\,\hat{u}(z) \end{aligned}$$

Transfer Functions

Well known technique in system theory: the z-**transform**. Replace a sequence $(x_n)_{n\in\mathbb{N}_0}$ by a function

$$\sum_{n=0}^{\infty} x_n\, z^n \quad =: \hat{x}(z)$$

Then if $x(0) = 0$

$$\begin{aligned} z^{-1}\,\hat{x}(z) &= A\,\hat{x}(z) + B\,\hat{u}(z) \\ \hat{y}(z) &= C\,\hat{x}(z) + D\,\hat{u}(z) \end{aligned}$$

Now eliminate x and obtain a direct **input-output relation**

$$\hat{y}(z) = \Theta(z)\,\hat{u}(z)$$

with the socalled **transfer function**

$$\Theta(z) = D + C \sum_{n\in\mathbb{N}_0} A^n B\, z^{n+1}$$

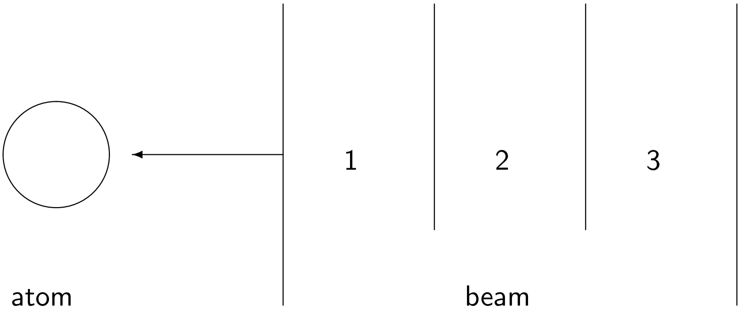
Many properties of the system are encoded in Θ in a nice way.

Toy Model

We want to discuss a new approach to introduce a similar tool for quantum mechanical systems.

Toy Model

We want to discuss a new approach to introduce a similar tool for quantum mechanical systems.



Interactions

Given

three Hilbert spaces $\mathcal{H}$, $\mathcal{K}$, $\mathcal{P}$

a unitary operator $U : \mathcal{H} \otimes \mathcal{K} \to \mathcal{H} \otimes \mathcal{P}$
$(U^* U = UU^* = \mathbb{1})$

unit vectors $\Omega^{\mathcal{H}} \in \mathcal{H}$, $\Omega^{\mathcal{K}} \in \mathcal{K}$, $\Omega^{\mathcal{P}} \in \mathcal{P}$ such that

$$U \left(\Omega^{\mathcal{H}} \otimes \Omega^{\mathcal{K}}\right) = \Omega^{\mathcal{H}} \otimes \Omega^{\mathcal{P}}$$

we call U an **interaction** with **vacuum vectors** $\Omega^{\mathcal{H}}, \Omega^{\mathcal{K}}, \Omega^{\mathcal{P}}$.

Repeated Interaction 1

Infinite Hilbert space tensor products

$$\mathcal{K}_\infty := \bigotimes_{\ell=1}^{\infty} \mathcal{K}_\ell \qquad \mathcal{K}_\ell \simeq \mathcal{K}$$

$$\mathcal{P}_\infty := \bigotimes_{\ell=1}^{\infty} \mathcal{P}_\ell \qquad \mathcal{P}_\ell \simeq \mathcal{P}$$

along unit vectors $\Omega_\infty^{\mathcal{K}} = \bigotimes_1^\infty \Omega^{\mathcal{K}}$ and $\Omega_\infty^{\mathcal{P}} = \bigotimes_1^\infty \Omega^{\mathcal{P}}$.

Repeated Interaction 1

Infinite Hilbert space tensor products

$$\mathcal{K}_\infty := \bigotimes_{\ell=1}^{\infty} \mathcal{K}_\ell \qquad \mathcal{K}_\ell \simeq \mathcal{K}$$

$$\mathcal{P}_\infty := \bigotimes_{\ell=1}^{\infty} \mathcal{P}_\ell \qquad \mathcal{P}_\ell \simeq \mathcal{P}$$

along unit vectors $\Omega_\infty^{\mathcal{K}} = \bigotimes_1^\infty \Omega^{\mathcal{K}}$ and $\Omega_\infty^{\mathcal{P}} = \bigotimes_1^\infty \Omega^{\mathcal{P}}$.

natural embeddings

$$\mathcal{H} \simeq \mathcal{H} \otimes \Omega_\infty^{\mathcal{K}} \subset \mathcal{H} \otimes \mathcal{K}_\infty \supset \Omega^{\mathcal{H}} \otimes \mathcal{K}_\infty \simeq \mathcal{K}_\infty .$$

Repeated Interaction 2

We can now define repeated interactions. For $\ell \in \mathbb{N}$ let

$$U_\ell : \mathcal{H} \otimes \mathcal{K}_\infty \to \mathcal{H} \otimes \mathcal{K}_{[1,\ell-1]} \otimes \mathcal{P}_\ell \otimes \mathcal{K}_{[\ell+1,\infty)}$$

be the unitary operator which is equal to U on $\mathcal{H} \otimes \mathcal{K}_\ell$ and which acts identically on the other factors of the tensor product.

Repeated Interaction 2

We can now define repeated interactions. For $\ell \in \mathbb{N}$ let

$$U_\ell : \mathcal{H} \otimes \mathcal{K}_\infty \to \mathcal{H} \otimes \mathcal{K}_{[1,\ell-1]} \otimes \mathcal{P}_\ell \otimes \mathcal{K}_{[\ell+1,\infty)}$$

be the unitary operator which is equal to U on $\mathcal{H} \otimes \mathcal{K}_\ell$ and which acts identically on the other factors of the tensor product.
The **repeated interaction** up to time $n \in \mathbb{N}$ is defined by

$$U(n) := U_n \dots U_1 : \mathcal{H} \otimes \mathcal{K}_\infty \to \mathcal{H} \otimes \mathcal{P}_{[1,n]} \otimes \mathcal{K}_{[n+1,\infty)}$$

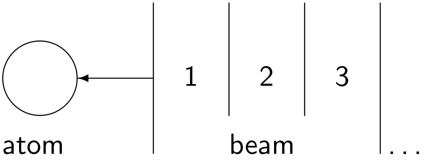
Repeated Interaction 2

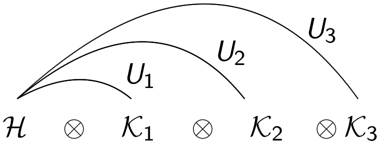
We can now define repeated interactions. For $\ell \in \mathbb{N}$ let

$$U_\ell : \mathcal{H} \otimes \mathcal{K}_\infty \to \mathcal{H} \otimes \mathcal{K}_{[1,\ell-1]} \otimes \mathcal{P}_\ell \otimes \mathcal{K}_{[\ell+1,\infty)}$$

be the unitary operator which is equal to U on $\mathcal{H} \otimes \mathcal{K}_\ell$ and which acts identically on the other factors of the tensor product.
The **repeated interaction** up to time $n \in \mathbb{N}$ is defined by

$$U(n) := U_n \dots U_1 : \mathcal{H} \otimes \mathcal{K}_\infty \to \mathcal{H} \otimes \mathcal{P}_{[1,n]} \otimes \mathcal{K}_{[n+1,\infty)}$$





Markov Process

We can think of our model as a **noncommutative Markov chain** or, from a physicist's point of view, as a Markovian approximation of a repeated atom-field interaction.

Markov Process

We can think of our model as a **noncommutative Markov chain** or, from a physicist's point of view, as a Markovian approximation of a repeated atom-field interaction.

Change of an observable $X \in \mathcal{B}(\mathcal{H})$ until time n compressed to $\mathcal{H}$:

$$Z_n(X) = P_{\mathcal{H}}\, U(n)^*\, X \otimes 1\, U(n)|_{\mathcal{H}}.$$

Markov Process

We can think of our model as a **noncommutative Markov chain** or, from a physicist's point of view, as a Markovian approximation of a repeated atom-field interaction.
Change of an observable $X \in \mathcal{B}(\mathcal{H})$ until time n compressed to $\mathcal{H}$:

$$Z_n(X) = P_{\mathcal{H}}\, U(n)^*\, X \otimes 1\, U(n)|_{\mathcal{H}}.$$

For ONB (ϵ_j) of the Hilbert space $\mathcal{P}$ and for $\xi \in \mathcal{H}$ write

$$U(\xi \otimes \Omega^{\mathcal{K}}) = \sum_j A_j \xi \otimes \epsilon_j$$

with operators $A_j \in \mathcal{B}(\mathcal{H})$. Then

$$Z_n(X) = \sum_{j_1, j_2, \dots, j_n} A_{j_1}^* \dots A_{j_n}^*\, X\, A_{j_n} \dots A_{j_1} = Z^n(X),$$

where $Z = \sum_j A_j^* \cdot A_j : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is a noncommutative **transition operator**: semigroup property of Markov processes.

Example 1

Example 1.

$$\mathcal{H} = \mathcal{K} = \mathcal{P} = \mathbb{C}^2, \quad 0 < p < 1$$

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{1-p} & -\sqrt{p} & 0 \\ 0 & \sqrt{p} & \sqrt{1-p} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Interpret the two basis vectors as "empty" and "occupied". Then the interaction describes a photon changing to a free place with probability p.

Example 2.

(discrete) Jaynes-Cummings model

$$\mathcal{H} = \ell^2(\mathbb{N}_0), \quad \mathcal{K} = \mathcal{P} = \mathbb{C}^2$$

$$\begin{aligned} U\,|0,0> &:= |0,0> \\ U\,|n-1,1> &:= \alpha_n\,|n-1,1> + \beta_n\,|n,0> \;\; (\textit{absorption}) \\ U\,|n,0> &:= \gamma_n\,|n-1,1> + \delta_n\,|n,0> \;\; (\textit{spontan. emission}) \end{aligned}$$

$$\text{with } \begin{pmatrix} \alpha_n & \beta_n \\ \gamma_n & \delta_n \end{pmatrix} \text{ unitary}, \quad n \in \mathbb{N}$$

Some Concepts from Multivariate Operator Theory

$T_1, \ldots, T_d \in \mathcal{B}(\mathcal{L})$ for a Hilbert space $\mathcal{L}$ $\quad$ ($d = \infty$ allowed)

Some Concepts from Multivariate Operator Theory

$T_1, \ldots, T_d \in \mathcal{B}(\mathcal{L})$ for a Hilbert space $\mathcal{L}$ ($d = \infty$ allowed)

$\underline{T} = (T_1, \ldots, T_d)$ is called a **row contraction** if it is contractive as an operator from $\bigoplus_1^d \mathcal{L}$ to $\mathcal{L}$ or, equivalently, if $\sum_1^d T_j T_j^* \leq 1$.

Some Concepts from Multivariate Operator Theory

$T_1, \ldots, T_d \in \mathcal{B}(\mathcal{L})$ for a Hilbert space $\mathcal{L}$ $\quad (d = \infty$ allowed)

$\underline{T} = (T_1, \ldots, T_d)$ is called a **row contraction** if it is contractive as an operator from $\bigoplus_1^d \mathcal{L}$ to $\mathcal{L}$ or, equivalently, if $\sum_1^d T_j T_j^* \leq 1$.

$\underline{T} = (T_1, \ldots, T_d)$ is called a **row isometry** if it is isometric as an operator from $\bigoplus_1^d \mathcal{L}$ to $\mathcal{L}$ or, equivalently, if the T_j are isometries with orthogonal ranges.

Some Concepts from Multivariate Operator Theory

$T_1, \dots, T_d \in \mathcal{B}(\mathcal{L})$ for a Hilbert space $\mathcal{L}$ $\quad (d = \infty$ allowed)

$\underline{T} = (T_1, \dots, T_d)$ is called a **row contraction** if it is contractive as an operator from $\bigoplus_1^d \mathcal{L}$ to $\mathcal{L}$ or, equivalently, if $\sum_1^d T_j T_j^* \leq 1$.

$\underline{T} = (T_1, \dots, T_d)$ is called a **row isometry** if it is isometric as an operator from $\bigoplus_1^d \mathcal{L}$ to $\mathcal{L}$ or, equivalently, if the T_j are isometries with orthogonal ranges.

A row isometry $\underline{T} = (T_1, \dots, T_d)$ is called a **row shift** if there exists a subspace $\mathcal{E}$ of $\mathcal{L}$ (the wandering subspace) such that $\mathcal{L} = \bigoplus_{\alpha \in F_d^+} T_\alpha \mathcal{E}$ $\quad (F_d^+$ free semigroup with generators $1, \dots, d)$

Outgoing Cuntz Scattering System

An **outgoing Cuntz scattering system** is a collection

$$\left(\mathcal{L},\, \underline{V} = (V_1, \ldots, V_d),\, \mathcal{G}_*^+,\, \mathcal{G}\right)$$

where $\underline{V}$ is a row isometry on the Hilbert space $\mathcal{L}$ and $\mathcal{G}_*^+$ and $\mathcal{G}$ are subspaces of $\mathcal{L}$ such that

Outgoing Cuntz Scattering System

An **outgoing Cuntz scattering system** is a collection

$$\big(\mathcal{L},\ \underline{V} = (V_1, \ldots, V_d),\ \mathcal{G}_*^+,\ \mathcal{G}\big)$$

where $\underline{V}$ is a row isometry on the Hilbert space $\mathcal{L}$ and $\mathcal{G}_*^+$ and $\mathcal{G}$ are subspaces of $\mathcal{L}$ such that

1. $\mathcal{G}_*^+$ is the smallest $\underline{V}$-invariant subspace containing

   $$\mathcal{E}_* := \mathcal{L} \ominus span_{j=1,\ldots,d}\ V_j \mathcal{L}\ ,$$

   thus $\underline{V}|_{\mathcal{G}_*^+}$ is a row shift and $\mathcal{G}_*^+ = \bigoplus_{\alpha \in F_d^+} V_\alpha \mathcal{E}_*$ (shift part of $\underline{V}$ in Wold decomposition)

Outgoing Cuntz Scattering System

An **outgoing Cuntz scattering system** is a collection

$$\big(\mathcal{L},\, \underline{V} = (V_1, \dots, V_d),\, \mathcal{G}_*^+,\, \mathcal{G}\big)$$

where $\underline{V}$ is a row isometry on the Hilbert space $\mathcal{L}$ and $\mathcal{G}_*^+$ and $\mathcal{G}$ are subspaces of $\mathcal{L}$ such that

1. $\mathcal{G}_*^+$ is the smallest $\underline{V}$-invariant subspace containing

   $$\mathcal{E}_* := \mathcal{L} \ominus span_{j=1,\dots,d}\ V_j \mathcal{L}\ ,$$

   thus $\underline{V}|_{\mathcal{G}_*^+}$ is a row shift and $\mathcal{G}_*^+ = \bigoplus_{\alpha \in F_d^+} V_\alpha \mathcal{E}_*$ (shift part of $\underline{V}$ in Wold decomposition)
2. $\underline{V}|_{\mathcal{G}}$ is a row shift, thus $\mathcal{G} = \bigoplus_{\alpha \in F_d^+} V_\alpha \mathcal{E}$ with

   $$\mathcal{E} := \mathcal{G} \ominus span_{j=1,\dots,d}\ V_j \mathcal{G}.$$

Outgoing Cuntz Scattering System - Reference

Cuntz scattering systems have been introduced in

J. Ball, V. Vinnikov

Lax-Phillips Scattering and Conservative Linear Systems:
A Cuntz-Algebra Multidimensional Setting.
Memoirs AMS, vol. 178 (2005)

Outgoing Cuntz Scattering System - Reference

Cuntz scattering systems have been introduced in

J. Ball, V. Vinnikov

Lax-Phillips Scattering and Conservative Linear Systems:
A Cuntz-Algebra Multidimensional Setting.
Memoirs AMS, vol. 178 (2005)

In this paper the emphasis is on generalizing ideas from Lax-Phillips scattering to a multivariate operator setting.
We want to make the connection with quantum probability.

Outgoing Cuntz Scat.System from Interaction Model 1

Theorem:
Let U be an interaction with vacuum vectors $\Omega^{\mathcal{H}}$, $\Omega^{\mathcal{K}}$, $\Omega^{\mathcal{P}}$. Then we have an outgoing Cuntz scattering system

$$(\mathcal{H} \otimes \mathcal{K}_\infty)^o, \ \underline{V} = (V_1, \ldots, V_d), \ \mathcal{G}_*^+, \ \mathcal{G}$$

where

Outgoing Cuntz Scat.System from Interaction Model 1

Theorem:

Let U be an interaction with vacuum vectors $\Omega^{\mathcal{H}}$, $\Omega^{\mathcal{K}}$, $\Omega^{\mathcal{P}}$. Then we have an outgoing Cuntz scattering system

$$(\mathcal{H} \otimes \mathcal{K}_\infty)^o,\ \underline{V} = (V_1, \ldots, V_d),\ \mathcal{G}_*^+,\ \mathcal{G}$$

where

$$(\mathcal{H} \otimes \mathcal{K}_\infty)^o := (\mathcal{H} \otimes \mathcal{K}_\infty) \ominus \mathbb{C}(\Omega^{\mathcal{H}} \otimes \Omega^{\mathcal{K}}_\infty)$$

(orthogonal complement of the vacuum)

Outgoing Cuntz Scat.System from Interaction Model 1

Theorem:

Let U be an interaction with vacuum vectors $\Omega^{\mathcal{H}}$, $\Omega^{\mathcal{K}}$, $\Omega^{\mathcal{P}}$. Then we have an outgoing Cuntz scattering system

$$(\mathcal{H} \otimes \mathcal{K}_\infty)^o,\ \underline{V} = (V_1, \ldots, V_d),\ \mathcal{G}_*^+,\ \mathcal{G}$$

where

$$(\mathcal{H} \otimes \mathcal{K}_\infty)^o := (\mathcal{H} \otimes \mathcal{K}_\infty) \ominus \mathbb{C}(\Omega^{\mathcal{H}} \otimes \Omega^{\mathcal{K}}_\infty)$$

(orthogonal complement of the vacuum)

$$V_j\big(\xi \otimes \eta\big) := U^*(\xi \otimes \epsilon_j) \otimes \eta \in (\mathcal{H} \otimes \mathcal{K}_1) \otimes \mathcal{K}_{[2,\infty)}$$

for $\xi \in \mathcal{H}$ and $\eta \in \mathcal{K}_\infty$ and $\big(\epsilon_j\big)$ an ONB of $\mathcal{P}$

Outgoing Cuntz Scat.System from Interaction Model 2

Wold decomposition

$$\mathcal{E}_* = U_1^* \mathcal{Y} \subset \mathcal{H} \otimes \mathcal{K}_1, \quad \mathcal{G}_*^+ = \bigoplus_{\alpha \in F_d^+} V_\alpha \mathcal{E}_*$$

$$\text{with } \mathcal{Y} := \Omega^{\mathcal{H}} \otimes (\Omega_1^{\mathcal{P}})^\perp \otimes \Omega_{[2,\infty)} \subset \mathcal{P}_\infty^o$$

Outgoing Cuntz Scat.System from Interaction Model 2

Wold decomposition

$$\mathcal{E}_* = U_1^* \mathcal{Y} \subset \mathcal{H} \otimes \mathcal{K}_1, \quad \mathcal{G}_*^+ = \bigoplus_{\alpha \in F_d^+} V_\alpha \mathcal{E}_*$$

$$\text{with } \mathcal{Y} := \Omega^{\mathcal{H}} \otimes (\Omega_1^{\mathcal{P}})^\perp \otimes \Omega_{[2,\infty)} \subset \mathcal{P}_\infty^o$$

For the second row shift we take

$$\mathcal{E} := \mathcal{H} \otimes (\Omega_1^{\mathcal{K}})^\perp \otimes \Omega_{[2,\infty)}^{\mathcal{K}}, \quad \mathcal{G} = \bigoplus_{\alpha \in F_d^+} V_\alpha \mathcal{E}.$$

Outgoing Cuntz Scat.System from Interaction Model 3

- The Wold decomposition is very **explicit** here.

Outgoing Cuntz Scat.System from Interaction Model 3

- The Wold decomposition is very **explicit** here.

- $\underline{V} = (V_1, \ldots, V_d)$ is an **isometric dilation** (in the sense of Popescu) of the row contraction $(A_1^*, \ldots, A_d^*)$ appearing in the noncommutative transition operator. As it is written it is usually not minimal but

Outgoing Cuntz Scat.System from Interaction Model 3

- The Wold decomposition is very **explicit** here.
- $\underline{V} = (V_1, \ldots, V_d)$ is an **isometric dilation** (in the sense of Popescu) of the row contraction $(A_1^*, \ldots, A_d^*)$ appearing in the noncommutative transition operator. As it is written it is usually not minimal but
- the setting relates more directly to **physical models**.

F_d^+-Linear Systems – Input and Output

- **input space** $\mathcal{U} := \mathcal{E} = \mathcal{H} \otimes (\Omega_1^{\mathcal{K}})^\perp \otimes \Omega_{[2,\infty)}^{\mathcal{K}} \quad \subset (\mathcal{H} \otimes \mathcal{K}_\infty)^o$,

F_d^+-Linear Systems – Input and Output

- **input space** $\mathcal{U} := \mathcal{E} = \mathcal{H} \otimes (\Omega_1^{\mathcal{K}})^\perp \otimes \Omega_{[2,\infty)}^{\mathcal{K}} \quad \subset (\mathcal{H} \otimes \mathcal{K}_\infty)^o$,
- **output space** $\mathcal{Y} := (\Omega_1^{\mathcal{P}})^\perp \otimes \Omega_{[2,\infty)}^{\mathcal{P}} \quad \subset (\mathcal{P}_\infty)^o$

F_d^+-Linear Systems – Input and Output

- **input space** $\mathcal{U} := \mathcal{E} = \mathcal{H} \otimes (\Omega_1^{\mathcal{K}})^\perp \otimes \Omega_{[2,\infty)}^{\mathcal{K}} \quad \subset (\mathcal{H} \otimes \mathcal{K}_\infty)^o$,
- **output space** $\mathcal{Y} := (\Omega_1^{\mathcal{P}})^\perp \otimes \Omega_{[2,\infty)}^{\mathcal{P}} \quad \subset (\mathcal{P}_\infty)^o$

With $H \otimes \mathcal{K} = \mathcal{H} \oplus \mathcal{U}$ the interaction U maps $\mathcal{H} \oplus \mathcal{U}$ onto $\mathcal{H} \otimes \mathcal{P}$ which contains $\mathcal{Y}$ (identifying $\mathcal{P}$ and $\mathcal{P}_1$). Hence for $j = 1, \ldots, d$ we can define

$$A_j : \mathcal{H} \to \mathcal{H}, \quad B_j : \mathcal{U} \to \mathcal{H}, \quad C : \mathcal{H} \to \mathcal{Y}, \quad D : \mathcal{U} \to \mathcal{Y}$$

$$\begin{aligned} U(\xi \oplus \eta) &=: \sum_{j=1}^{d} \left(A_j \xi + B_j \eta\right) \otimes \epsilon_j \\ P_{\mathcal{Y}}\, U(\xi \oplus \eta) &=: C\xi + D\eta, \end{aligned}$$

with $\xi \in \mathcal{H}$, $\eta \in \mathcal{U}$ and $\left(\epsilon_j\right)_{j=1}^{d}$ ONB of $\mathcal{P}$ and $P_{\mathcal{Y}}$ proj. onto $\mathcal{Y}$

F_d^+-Linear systems – Colligations

Further we define the **colligation**

$$\mathcal{C}_U := \begin{pmatrix} A_1 & B_1 \\ \vdots & \vdots \\ A_d & B_d \\ C & D \end{pmatrix} : \quad \mathcal{H} \oplus \mathcal{U} \to \bigoplus_{j=1}^{d} \mathcal{H} \oplus \mathcal{Y}$$

F_d^+-Linear systems – Colligations

Further we define the **colligation**

$$\mathcal{C}_U := \begin{pmatrix} A_1 & B_1 \\ \vdots & \vdots \\ A_d & B_d \\ C & D \end{pmatrix} : \quad \mathcal{H} \oplus \mathcal{U} \to \bigoplus_{j=1}^{d} \mathcal{H} \oplus \mathcal{Y}$$

The colligation $\mathcal{C}_U$ gives rise to a F_d^+-**linear system** Σ_U (noncommutative Fornasini-Marchesini system)

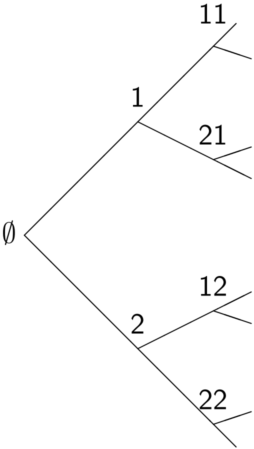
$$\begin{aligned} x(j\alpha) &= A_j\, x(\alpha) + B_j\, u(\alpha) \\ y(\alpha) &= C\, x(\alpha) + D\, u(\alpha), \end{aligned}$$

where $j = 1, \ldots, d$, further α, $j\alpha$ (concatenation) are words in F_d^+ and

$$x : F_d^+ \to \mathcal{H}, \quad u : F_d^+ \to \mathcal{U}, \quad y : F_d^+ \to \mathcal{Y}.$$

F_d^+-Linear Systems – Example

Given $x(\emptyset)$ and u we can use Σ_U to compute x and y recursively.



$\ldots$

dyadic tree for $d = 2$

Input - Output Relation

Can we describe an F_d^+-linear system by a transfer function?

Input - Output Relation

Can we describe an F_d^+-linear system by a transfer function?
For this we define the **noncommutative** z**-transform** of x as

$$\hat{x}(z) = \sum_{\alpha \in F_d^+} x(\alpha) z^\alpha,$$

where $z^\alpha = z_{\alpha_n} \dots z_{\alpha_1}$ if $\alpha = \alpha_n \dots \alpha_1 \in F_d^+$ and $z = (z_1, \dots, z_d)$ is a d-tuple of formal non-commuting indeterminates. Similarly $\hat{u}(z) = \sum_{\alpha \in F_d^+} u(\alpha) z^\alpha$ and $\hat{y}(z) = \sum_{\alpha \in F_d^+} y(\alpha) z^\alpha$.

Input - Output Relation

Can we describe an F_d^+-linear system by a transfer function?
For this we define the **noncommutative z-transform** of x as

$$\hat{x}(z) = \sum_{\alpha \in F_d^+} x(\alpha) z^\alpha,$$

where $z^\alpha = z_{\alpha_n} \dots z_{\alpha_1}$ if $\alpha = \alpha_n \dots \alpha_1 \in F_d^+$ and $z = (z_1, \dots, z_d)$ is a d-tuple of formal non-commuting indeterminates. Similarly $\hat{u}(z) = \sum_{\alpha \in F_d^+} u(\alpha) z^\alpha$ and $\hat{y}(z) = \sum_{\alpha \in F_d^+} y(\alpha) z^\alpha$.
For $x(\emptyset) = 0$ we have the **input-output relation**

$$\hat{y}(z) = \Theta_U(z)\, \hat{u}(z)$$

where

$$\Theta_U(z) := \sum_{\alpha \in F_d^+} \Theta_U^{(\alpha)} z^\alpha := D + C \sum_{\substack{\beta \in F_d^+ \\ j=1,\dots,d}} A_\beta B_j z^{\beta j}$$

Noncommutative Transfer Function

We call the formal non-commutative power series $\Theta_U(z) := \sum_{\alpha \in F_d^+} \Theta_U^{(\alpha)} z^\alpha$ the (noncommutative) **transfer function** associated to the interaction U. The 'Taylor coefficients' $\Theta_U^{(\alpha)}$ are operators from $\mathcal{U}$ to $\mathcal{Y}$.

Noncommutative Transfer Function

We call the formal non-commutative power series $\Theta_U(z) := \sum_{\alpha \in F_d^+} \Theta_U^{(\alpha)} z^\alpha$ the (noncommutative) **transfer function** associated to the interaction U. The 'Taylor coefficients' $\Theta_U^{(\alpha)}$ are operators from $\mathcal{U}$ to $\mathcal{Y}$.
We can proceed from formal power series to operators between Hilbert spaces.

Theorem

The input-output relation

$$\hat{y}(z) = \Theta_U(z)\, \hat{u}(z)$$

corresponds to a **contraction**

$$M_{\Theta_U} : \ell^2(F_d^+, \mathcal{U}) \to \ell^2(F_d^+, \mathcal{Y})$$

which (with $x(\emptyset) = 0$) maps an input sequence u to the corresponding output sequence y.

Multi-Analytic Operators and Noncommutative Schur Class

The operator M_{Θ_U} has the property that it intertwines with right translation, i.e., for all $j = 1, \ldots, d$

$$M_{\Theta_U}\Big(\sum_{\alpha \in F_d^+} x(\alpha) z^\alpha \, z^j\Big) \;=\; M_{\Theta_U}\Big(\sum_{\alpha \in F_d^+} x(\alpha) z^\alpha\Big)\, z^j \; .$$

Such operators have been called **analytic intertwining operators** or **multianalytic operators**: there are analogies to the theory of multiplication operators by analytic functions on Hardy spaces. The non-commutative power series Θ_U is called the **symbol** of M_{Θ_U}.

Multi-Analytic Operators and Noncommutative Schur Class

The operator M_{Θ_U} has the property that it intertwines with right translation, i.e., for all $j = 1, \ldots, d$

$$M_{\Theta_U}\Big(\sum_{\alpha \in F_d^+} x(\alpha) z^\alpha \, z^j \Big) \;=\; M_{\Theta_U}\Big(\sum_{\alpha \in F_d^+} x(\alpha) z^\alpha \Big) \, z^j \; .$$

Such operators have been called **analytic intertwining operators** or **multianalytic operators**: there are analogies to the theory of multiplication operators by analytic functions on Hardy spaces. The non-commutative power series Θ_U is called the **symbol** of M_{Θ_U}.

It was one of the motivations for this work to make this theory available for the study of interaction models and non-commutative Markov chains. Note that because M_{Θ_U} is a contraction the transfer function Θ_U belongs to the socalled **non-commutative Schur class** $S_{nc,d}(\mathcal{U}, \mathcal{Y})$.

Physical Interpretation – Input

We may think of $\mathcal{H}$ as the (quantum mechanical) Hilbert space of an atom, $\mathcal{K}_\ell$ as the Hilbert space of a part of a light beam or field which interacts with the atom at time ℓ.

Then we think of $\Omega^{\mathcal{H}}$ as a vacuum state of the atom and of $\Omega^{\mathcal{K}} = \Omega^{\mathcal{P}}$ in $\mathcal{K} = \mathcal{P}$ as a state indicating that **no photon** is present.

Physical Interpretation – Input

We may think of $\mathcal{H}$ as the (quantum mechanical) Hilbert space of an atom, $\mathcal{K}_\ell$ as the Hilbert space of a part of a light beam or field which interacts with the atom at time ℓ.

Then we think of $\Omega^{\mathcal{H}}$ as a vacuum state of the atom and of $\Omega^{\mathcal{K}} = \Omega^{\mathcal{P}}$ in $\mathcal{K} = \mathcal{P}$ as a state indicating that **no photon** is present.

- The **input**

$$\eta \in \mathcal{U} = \mathcal{H} \otimes (\Omega_1^{\mathcal{K}})^\perp \otimes \Omega_{[2,\infty)}^{\mathcal{K}} \subset \mathcal{H} \otimes \mathcal{K}_\infty$$

represents a vector state with

- **photons arriving at time** 1 and stimulating an interaction between the atom and the field,
- but no further photons arriving at later times.
- Nevertheless it may happen that some activity (emission) is induced which goes on for a longer period.

Physical Interpretation – Output

The orthogonal projection P_α onto

$$\epsilon_{\alpha_1} \otimes \ldots \otimes \epsilon_{\alpha_{n-1}} \otimes (\Omega_n^{\mathcal{P}})^\perp \otimes \Omega_{[n+1,\infty)},$$

corresponds to the following **event**:

- We measure data $\alpha_1, \ldots, \alpha_{n-1}$ at times $1, \ldots, n-1$ in the field, finally there is a last detection of photons corresponding to $(\Omega_n^{\mathcal{P}})^\perp$ at time n, nothing happens after time n.
- This experimental record is obtained by **measuring** (at times indexed by the positive integers) **an observable** $Y \in \mathcal{B}(\mathcal{P})$ with eigenvectors $\epsilon_1, \ldots, \epsilon_d$. Such lists of data have been used for indirect measurements of an atom, for quantum filtering and for updating protocols such as quantum trajectories.

Physical Interpretation of Taylor Coefficients

We can obtain the following **formula for the Taylor coefficients**

$$P_\alpha \, U(n)\eta \; = \; \Theta_U^{(\alpha)}\eta$$

According to the usual probabilistic interpretation of quantum mechanics this means for example that

$$\pi_\alpha := \| \, \Theta_U^{(\alpha)}\eta \, \|^2$$

is the probability for the event described by P_α if we start in the state η at time 0.

- Actually the transfer function also keeps track of the complex amplitudes and contains additional coherent information.

Physical Interpretation of Taylor Coefficients

We can obtain the following **formula for the Taylor coefficients**

$$P_\alpha \, U(n)\eta \;=\; \Theta_U^{(\alpha)}\eta$$

According to the usual probabilistic interpretation of quantum mechanics this means for example that

$$\pi_\alpha := \| \, \Theta_U^{(\alpha)}\eta \, \|^2$$

is the probability for the event described by P_α if we start in the state η at time 0.

- Actually the transfer function also keeps track of the complex amplitudes and contains additional coherent information.

Conclusion: We can think of the transfer function Θ_U as a convenient way to assemble such data into a **single mathematical object**.

Observability and Scattering Theory

- The control theoretic concept of **'observability'** for our model is closely related to an operator-algebraic **scattering theory** for noncommutative Markov chains

  (as in B. Kümmerer, H. Maassen, A Scattering Theory for Markov Chains. IDAQP vol.**3** (2000), 161-176)

Observability and Scattering Theory

- The control theoretic concept of **'observability'** for our model is closely related to an operator-algebraic **scattering theory** for noncommutative Markov chains

  (as in B. Kümmerer, H. Maassen, A Scattering Theory for Markov Chains. IDAQP vol.**3** (2000), 161-176)

- Roughly: A system is called observable if by studying the outputs for given inputs we can determine the internal state of the system.

  In our model: We observe output fields for given input fields and we want to determine the state of the atom from that.

  If a system is asymptotically complete in the sense of scattering theory then this can be done. This is the link!

Observability Operator

Guided by such considerations, in our setting this can be made precise. We define the **observability operator**

$$\begin{aligned} W_O : \mathcal{H} &\to \ell^2(F_d^+, \mathcal{Y}) \\ \xi &\mapsto \big(C\, A_\alpha\, \xi\big)_{\alpha \in F_d^+} \end{aligned}$$

If W_O is **injective** then the system is called **observable**. This is the mathematical counterpart of our intuitive discussion above.

Observability and Scattering Theory – Main Result

For simplicity we state the following Theorem for finite-dimensional systems only. But most of the assertions are true in general under technical modifications.

Theorem:

The following are **equivalent**:

- The system is **observable**.
- The **observability operator** is **isometric**.
- The transfer function Θ_U is **inner**, i.e., the associated multi-analytic operator M_{Θ_U} is isometric.
- The noncommutative transition operator $Z : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is **ergodic** (i.e., the fixed point space is trivial)
- We have **asymptotic completeness** in (a suitable version of) Kümmerer-Maassen scattering theory.

Open Ends

The classical transfer function plays an important role in **control theory**. Hence we expect the noncommutative transfer function to play its role in **quantum control**.

We have already seen that it relates to filtering.

Open Ends

The classical transfer function plays an important role in **control theory**. Hence we expect the noncommutative transfer function to play its role in **quantum control**.

We have already seen that it relates to filtering.

Another plan: Study **networks** of quantum systems. Are there effective ways to compute the transfer function of suitable networks consisting of many quantum systems?

Open Ends

The classical transfer function plays an important role in **control theory**. Hence we expect the noncommutative transfer function to play its role in **quantum control**.

We have already seen that it relates to filtering.

Another plan: Study **networks** of quantum systems. Are there effective ways to compute the transfer function of suitable networks consisting of many quantum systems?

Finally connections should appear to work already done for **continuous time models** (for example by Belavkin, Bouten, van Handel, James, Gough etc.).

Main Reference

For more details and for further references see

Rolf Gohm, Non-Commutative Markov Chains and Multi-Analytic Operators, Journal of Mathematical Analysis and Applications 364 (2010), 275-288 or arxiv:0902.3445

Related Work 1

- L. Bouten, R. van Handel, M. James, A Discrete Invitation to Quantum Filtering and Feedback Control. To appear in SIAM Review, arXiv:math/0606118
- J. Ball, V. Vinnikov, Lax-Phillips Scattering and Conservative Linear Systems: A Cuntz-Algebra Multidimensional Setting. Memoirs of the AMS, vol. 178, no. **837** (2005)
- S. Dey, R. Gohm, Characteristic Functions for Ergodic Tuples. Integral Equations and Operator Theory, **58** (2007), 43-63.
- S. Dey, R. Gohm, Characteristic Functions of Liftings. To appear in the Journal of Operator Theory, arXiv:0707.1417
- J.Gough, R. Gohm, M.Yanagisawa, Linear Quantum Feedback Networks. Phys. Rev. A **78**, 062104 (2008)

Related Work 2

- B. Kümmerer, H. Maassen, A Scattering Theory for Markov Chains. Inf. Dim. Analysis, Quantum Prob. and Related Topics, vol.**3** (2000), 161-176
- G. Popescu, Characteristic Functions for Infinite Sequences of Noncommuting Operators. J. Operator Theory **22** (1989), 51-71.
- G. Popescu, Multi-Analytic Operators on Fock Spaces. Math. Ann. **303** (1995), no. 1, 31-46.

Related Work 2

- B. Kümmerer, H. Maassen, A Scattering Theory for Markov Chains. Inf. Dim. Analysis, Quantum Prob. and Related Topics, vol.**3** (2000), 161-176
- G. Popescu, Characteristic Functions for Infinite Sequences of Noncommuting Operators. J. Operator Theory **22** (1989), 51-71.
- G. Popescu, Multi-Analytic Operators on Fock Spaces. Math. Ann. **303** (1995), no. 1, 31-46.

That's it. Thank you!

References 1

- D. Burgarth, V. Giovannetti, A Protocol for Cooling and Controlling Composite Systems by Local Interactions. Quantum Information and Many Body Quantum Systems, proceedings, M. Ericsson and S. Montangero (eds.), Pisa, Edizioni della Normale (2008)
- B.V.R. Bhat, An Index Theory for Quantum Dynamical Semigroups. Trans. Am. Math. Soc., **348** (1996), 561-583
- B.V.R. Bhat, Cocycles of CCR Flows. Mem. Am. Math. Soc., **709** (2001)
- J. Ball, G. Groenewald, T. Malakorn, Conservative Structured Noncommutative Multidimensional Linear Systems. D. Alpay, I. Gohberg (Eds.), The State Space Method, Generalizations and Applications, Operator Theory, Advances and Applications, Vol. **161**, Birkhäuser (2006)
- L. Bouten, R. van Handel, M. James, A Discrete Invitation to Quantum Filtering and Feedback Control. To appear in SIAM Review, arXiv:math/0606118
- J. Ball, V. Vinnikov, Lax-Phillips Scattering and Conservative Linear Systems: A Cuntz-Algebra Multidimensional Setting. Memoirs of the AMS, vol. 178, no. **837** (2005)
- S. Dey, R. Gohm, Characteristic Functions for Ergodic Tuples. Integral Equations and Operator Theory, **58** (2007), 43-63.
- S. Dey, R. Gohm, Characteristic Functions of Liftings. To appear in the Journal of Operator Theory, arXiv:0707.1417
- W. Feller, An Introduction to Probability Theory and its Applications. Vol. I, Wiley (1968)
- C. Foias, A.E. Frazho, I. Gohberg, M.A. Kaashoek, Metric Constrained Interpolation, Commutant Lifting and Systems. Operator Theory, Advances and Applications, Vol. **100**, Birkhäuser (1997)

References 2

- E. Fornasini, G. Marchesini, Doubly-indexed Dynamical Systems: State Space Models and Structural Properties. Math. Systems Theory **12** (1978), 59-72
- C. Gardiner, P. Zoller, Quantum Noise: A Handbook of Markovian and Non-Markovian Quantum Stochastic Methods with Applications to Quantum Optics. 2nd ed., Springer Series in Synergetics, Springer (2000)
- J.Gough, R. Gohm, M.Yanagisawa, Linear Quantum Feedback Networks. Phys. Rev. A **78**, 062104 (2008)
- J. Gough, M. James, Quantum Feedback Networks: Hamiltonian Formulation. To appear in Comm. Math. Phys., arXiv:0804.3442 (2008)
- R. Gohm, Noncommutative Stationary Processes. Springer LNM **1839** (2004)
- R. Gohm, B. Kümmerer, T. Lang, Noncommutative Symbolic Coding. Ergod.Th. & Dynam.Sys., **26** (2006), 1521-1548
- R.V. Kadison, J.R. Ringrose, Fundamentals of the Theory of Operator Algebras II. Academic Press (1983)
- B. Kümmerer, Quantum Markov Processes. A. Buchleitner, K. Hornberger (Eds.), Coherent Evolution in Noisy Environments, Springer LNP **611** (2002), 139-198
- B. Kümmerer, H. Maassen, A Scattering Theory for Markov Chains. Inf. Dim. Analysis, Quantum Prob. and Related Topics, vol.**3** (2000), 161-176
- B. Kümmerer, H. Maassen, Purification of Quantum Trajectories. Dynamics & Stochastics, Festschrift in honor of M.S. Keane, Lecture Notes-Monograph Series of the IMS, Vol. 48 (2006), 252-261

References 3

- G. Popescu, Isometric Dilations for Infinite Sequences of Noncommuting Operators. Trans. Amer. Math. Soc. **316** (1989), 523-536.
- G. Popescu, Multi-Analytic Operators on Fock Spaces. Math. Ann. **303** (1995), no. 1, 31-46.
- G. Popescu, Poisson Transforms on some C^*-Algebras generated by Isometries. J. Funct. Anal. **161** (1999), 27-61.
- G. Popescu, Free Holomorphic Functions on the Unit Ball of $\mathcal{B}(\mathcal{H})^n$. J. Funct. Anal. **241** (2006), 268-333.
- B. Sz.-Nagy, C. Foias, Harmonic Analysis of Operators. North Holland (1970)
- M. Yanagisawa, H. Kimura, Transfer Function Approach to Quantum Control, part I: Dynamics of Quantum Feedback Systems. IEEE Transactions on Automatic Control, **48** (2003), no. 12, 2107-2120
- M. Yanagisawa, H. Kimura, Transfer Function Approach to Quantum Control, part II: Control Concepts and Applications. IEEE Transactions on Automatic Control, **48** (2003), no. 12, 2121-2132