Hypercontractivity on the q-Araki-Woods algebras

Hun Hee Lee
Chungbuk National University

Joint work with Éric Ricard

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q-Fock space for $-1 \leq q \leq 1$ (Bożejko, Speicher '91)

- $\mathcal{H}_{\mathbb{R}}$: real Hilbert space, $\mathcal{H} = \mathcal{H}_{\mathbb{R}} + i\mathcal{H}_{\mathbb{R}}$: complexification. $\mathcal{F}_0(\mathcal{H}) = \mathbb{C}\Omega \oplus \bigoplus_{n\geq 1} \mathcal{H}^{\otimes n}$ (Free Fock space)
- P_n : Symmetrization operator on $\mathcal{H}^{\otimes n}$

$$P_0\Omega = \Omega, \quad P_n(f_1 \otimes \cdots \otimes f_n) = \sum_{\pi \in S_n} q^{i(\pi)} f_{\pi(1)} \otimes \cdots \otimes f_{\pi(n)},$$

  where S_n is the symmetric group on $\{1, 2, \cdots, n\}$ and $i(\pi) = \#\{(i,j) | 1 \leq i, j \leq n, \pi(i) > \pi(j)\}$.
- (q-**inner product**) $\langle \cdot, \cdot \rangle_q$ on $\mathcal{F}_0(\mathcal{H})$:

$$\langle \xi, \eta \rangle_q := \delta_{n,m} \langle \xi, P_n \eta \rangle_{\mathcal{H}} \text{ for } \xi \in \mathcal{H}^{\otimes n}, \eta \in \mathcal{H}^{\otimes m}.$$

- (q-**Fock space**) $\mathcal{F}_q(\mathcal{H}) := (\mathcal{F}_0(\mathcal{H}), \langle \cdot, \cdot \rangle_q)$, $-1 < q < 1$
  $\mathcal{F}_{\pm 1}(\mathcal{H}) := (\mathcal{F}_0(\mathcal{H}), \langle \cdot, \cdot \rangle_{\pm 1}) / \mathrm{Ker}(\langle \cdot, \cdot \rangle_{\pm 1})$.

q-generalized gaussian

- For $h \in \mathcal{H}$, $\ell_q(h)$ is the left creation operator defined by

$$\ell_q(h)\Omega = h,$$

$$\ell_q(h)h_1 \otimes \cdots \otimes h_n = h \otimes h_1 \otimes \cdots \otimes h_n$$

  for any $h_i \in \mathcal{H}, 1 \leq i \leq n$.
- $\ell_q^*(h)$ is the annihilation operator (the adjoint of $\ell_q(h)$).
- $(e_{\pm k})_{k\geq 1}$: an ONB of $\mathcal{H}_{\mathbb{R}}$.
  $\mu = (\mu_k)_{k\geq 1}$: a sequence of positive reals ≥ 1.
- (q-**generalized gaussian**)

$$g_{q,k} := \mu_k^{-1}\ell_q(e_k) + \mu_k \ell_q^*(e_{-k}).$$

q-Araki-woods algebra (Discrete Case)

- $\Gamma_q := \{g_{q,k}\}''_{k\geq 1} \subseteq B(\mathcal{F}_q(\mathcal{H})),\ \ -1 \leq q < 1$
  $\Gamma_1 := \{\exp(i \cdot g_{1,k})\}''_{k\geq 1} \subseteq B(\mathcal{F}_1(\mathcal{H}))$.
- $q = \pm 1$ Araki, Woods'69
  $q = 0$ Shlyakhtenko '97
  $-1 < q < 1$ Hiai '03.
- τ_q: The vacuum state defined by
  $\tau_q(X) = \langle X\Omega, \Omega\rangle_q$ for $X \in \Gamma_q$.
- τ_q is tracial when $\mu_k \equiv 1$, and actually Γ_q is a II_1 factor. When $\mu_k \neq 1$ for some k, then τ_q is not tracial anymore, and Γ_q is a type III VN-algebra.
- Ω is always a separating vector for Γ_q.

q-Ornstein-Uhlenbeck semigroup

- q-Ornstein-Uhlenbeck semigroup $P_t^q : \Gamma_q \to \Gamma_q$ is defined by
  $$P_t^q(X)\Omega = e^{-nt}(X\Omega) \text{ for any } X \in \Gamma_q \text{ with } X\Omega \in \mathcal{H}^{\otimes n}.$$
- For example, $P_t^q(g_{q,k_1} \cdots g_{q,k_n}) = e^{-nt} g_{q,k_1} \cdots g_{q,k_n}$.
- P_t^q is actually a semigroup of completely positive, normal, τ_q-preserving contractions that commute with the modular group of τ_q.

Hypercontractivity problem

- For which p, r and t does P_t^q extends to a contraction
$$P_t^q : L^p(\Gamma_q) \to L^r(\Gamma_q)?$$
- (**Nelson '73, classical case**) When $q = 1$ and $\mu \equiv 1$
$$\left\|P_t^q\right\|_{L^p \to L^r} \leq 1 \quad \Leftrightarrow e^{-2t} \leq \tfrac{p-1}{r-1}.$$
- (**Carlen, Lieb '93**) When $q = -1$ and $\mu \equiv 1$
$$\left\|P_t^q\right\|_{L^p \to L^r} \leq 1 \quad \Leftrightarrow e^{-2t} \leq \tfrac{p-1}{r-1}.$$
- (**Biane '97, tracial case**) When $-1 \leq q \leq 1$ and $\mu \equiv 1$
$$\left\|P_t^q\right\|_{L^p \to L^2} \leq 1 \quad \Leftrightarrow e^{-2t} \leq p-1, \ \ 1 < p \leq 2.$$
- **Question** : What happens for the non-tracial case? i.e. the case $\mu_k > 1$ for some k.

Non-commutative L^p spaces $1 \leq p < \infty$

- $\mathcal{N}$: VN-algebra with n.f. tracial state φ.
  $\|x\|_p := \varphi(|x|^p)^{\frac{1}{p}}$, $x \in \mathcal{N}$.
  $L^p(\mathcal{N}, \varphi) :=$ the completion of $(\mathcal{N}, \|\cdot\|_p)$.
- $\mathcal{M} \subseteq B(H)$: VN-algebra with n.f. state φ.
  $L^p(\mathcal{M}, \varphi)$ can be defined in the sense of Haagerup.
- Elements in $L^p(\mathcal{M}, \varphi)$ can be understood as (well-behaving) unbounded operators acting on a bigger Hilbert space $L^2(\mathbb{R}, H)$.
- There is a unique functional tr on $L^1(\mathcal{M}, \varphi)$ and a distinguished element $D_\varphi \in L^1(\mathcal{M}, \varphi)$ s.t. $\mathrm{tr}(xD_\varphi) = \varphi(x)$.
- $\mathcal{M}D_\varphi^{\frac{1}{p}}$ is a norm-dense subspace in $L^p(\mathcal{M}, \varphi)$.
  $x \in \mathcal{M} \overset{\text{identified}}{\longleftrightarrow} xD_\varphi^{\frac{1}{p}} \in L_p(\mathcal{M})$ in the sense of complex interpolation.

Extension to non-commutative L^p-setting

- $\mathcal{M}, \mathcal{N}$: VN-algebras with n.f. states φ and ψ, resp., with associated density operators D_φ and D_ψ.
  $T : \mathcal{M} \to \mathcal{N}$ be a c.p. contraction satisfying
  $$\begin{cases} \psi \circ T = \varphi \\ \sigma_t^\psi \circ T = T \circ \sigma_t^\varphi, \ \ \forall t \in \mathbb{R} \end{cases}.$$
- Let $1 \leq p, r < \infty$.

$$T^{p,r} : \mathcal{M} D_\varphi^{\frac{1}{p}} \to \mathcal{N} D_\psi^{\frac{1}{r}} \subseteq L_r(\mathcal{N}), \ \ x D_\varphi^{\frac{1}{p}} \mapsto (Tx) D_\psi^{\frac{1}{r}}.$$

- It is well-known that $T^{p,p}$ extends to a (complete) contraction.
- We will denote $T^{p,r}$ simply by T.

The main results

Theorem

Let $1 < p < 2$. Then, we have

$$\left\| P_t^q \right\|_{L^p \to L^2} \leq 1 \ \textit{ if } \ e^{-2t} \leq C\alpha_\mu^{4-\frac{8}{p}}(p-1),$$

where $\alpha_\mu = \sup_{n \geq 1} \mu_n$.

- Note that we get the same optimal order $p-1$ for $p \sim 1$.

Theorem

Suppose that $\alpha_\mu = \infty$, then P_t^q can not be extended to a contraction from $L^p(\Gamma_q)$ into $L^2(\Gamma_q)$ for any $1 \leq p < 2$.

Sketch of Proof

- For the "if" part, we approximate creations and annihilations (and consequently generalized gaussians) by using the **Baby Fock** model.
- P.A. Meyer '95: the case $q = 1$ or Bosonic case.
  P. Biane '97: the case $-1 \leq q \leq 1$.
  A. Nou '06: the case $-1 \leq q \leq 1$ focusing on the non-tracial case.
- For the "only if" part, we get the conclusion by examining 1-dimensional behavior as usual.

A. Nou's twisted baby Fock

- $I = \{\pm 1, \pm 2, \cdots, \pm n\}$
  $\varepsilon : I \times I \to \{\pm 1\}$: a choice of sign satisfying
  $\varepsilon(i,j) = \varepsilon(j,i)$, $\varepsilon(i,j) = \varepsilon(|i|, |j|)$, and $\varepsilon(i,i) = -1$.
- $\mathcal{A}(I, \varepsilon)$: unital algebra with generators $(x_i)_{i \in I}$ with
  $$x_i x_j - \varepsilon(i,j) x_j x_i = 2\delta_{i,j}, \quad i,j \in I.$$
  In particular, $x_i^2 = 1$.
  $\mathcal{A}(I, \varepsilon)$ is equipped with an involution $*$, such that $x_i^* = x_i$.
- $x_\emptyset := 1$ and $x_A := x_{i_1} \cdots x_{i_k}$ for $A = \{i_1 < \cdots < i_k\} \subseteq I$
  $\{x_A : A \subseteq I\}$ is a basis for $A(\mathcal{I}, \varepsilon)$, so that $\dim A(\mathcal{I}, \varepsilon) = 2^{2n}$.
- $\varphi^\varepsilon : \mathcal{A}(I, \varepsilon) \to \mathbb{C}$: a tracial state given by $\varphi^\varepsilon(x_A) = \delta_{A,\emptyset}$.
  $H = L^2(\mathcal{A}(I, \varepsilon), \varphi^\varepsilon)$: the corresponding L^2-space
  $\{x_A : A \subseteq I\}$: an ONB of H.

A. Nou's twisted baby Fock: continued

- (**Creations and Annihilations**)
$\beta_i^*(x_A) = \begin{cases} x_i x_A & \text{if } i \notin A \\ 0 & \text{if } i \in A \end{cases}$, $\beta_i(x_A) = \begin{cases} x_i x_A & \text{if } i \in A \\ 0 & \text{if } i \notin A \end{cases}$
- (**Gaussians and corresponding VN-algebra**)
$\gamma_i := \mu_i^{-1}\beta_i^* + \mu_i\beta_{-i}$, $1 \le i \le n$.
$\Gamma_n := \{\gamma_i : 1 \le i \le n\}'' \subseteq B(H) \cong M_{2^{2n}}$.
$\dim\Gamma_n = 2^n$.
$\tau^\varepsilon(\cdot) = \langle \cdot 1, 1\rangle$: the vacuum state.
- (**Relations satisfied by Gaussians**)

$$\begin{cases} \gamma_i\gamma_j - \varepsilon(i,j)\gamma_j\gamma_i = 0 & i \neq j \in I \\ \gamma_i^*\gamma_j - \varepsilon(i,j)\gamma_j\gamma_i^* = 0 & i \neq j \in I \\ \gamma_i^2 = (\gamma_i^*)^2 = 0 & i \in I \\ \gamma_i^*\gamma_i + \gamma_i\gamma_i^* = (\mu_i^2 + \mu_i^{-2})id & i \in I \end{cases}$$

A. Nou's twisted baby Fock: continued 2

- (ε-**Ornstein-Uhlenbeck semigroup**)
  $N_\varepsilon = \sum_{i \in I} \beta_i^* \beta_i$: the number operator on H.
  Then, for any $A \subseteq I$ we have $N_\varepsilon x_A = |A| \, x_A$.
  We define the ε-Ornstein-Uhlenbeck semigroup $P_t^\varepsilon : \Gamma_n \to \Gamma_n$ by
  $$P_t^\varepsilon(X)1 = e^{-tN_\varepsilon}(X1), \quad X \in \Gamma_n.$$
- P_t^ε is actually a semigroup of unital, completely positive maps.

Hypercontractivity of ε-Ornstein-Uhlenbeck semigroup

- Using Speicher's central limit procedure it is enough to show Hypercontractivity of ε-Ornstein-Uhlenbeck semigroup.
- Following the idea of Carlen/Lieb and Biane we use the induction on n, the number of generators.
- We start with any element $X \in \Gamma_n$, which is uniquely expressed as
$$X = a + \gamma_n b + \gamma_n^* c + y_n d,$$
where $a, b, c, d \in \Gamma_{n-1}$ and $y_n = \gamma_n^* \gamma_n - \mu_n^{-2} id$.
- We use y_n instead of $\gamma_n^* \gamma_n$ because of orthogonality.

Key estimates

- We will end up with the estimate

$$\left\| XD_n^{\frac{1}{p}} \right\|_p^2 \geq \left\| aD_{n-1}^{\frac{1}{p}} \right\|_p^2 + C_1(p-1) \left\| bD_{n-1}^{\frac{1}{p}} \right\|_p^2 + C_2(p-1) \left\| cD_{n-1}^{\frac{1}{p}} \right\|_p^2 + C_3(p-1) \left\| dD_{n-1}^{\frac{1}{p}} \right\|_p^2,$$

  where D_n and D_{n-1} are corresponding densities.
- (Optimal convexity inequality, Ball/Carlen/Lieb, '94)

$$\left(\frac{1}{2}\left[\, \|A+B\|_p^p + \|A-B\|_p^p \,\right]\right)^{\frac{2}{p}} \geq \|A\|_p^2 + (p-1)\,\|B\|_p^2, \quad A, B \in M_n.$$

- By the optimal convexity inequality and the fact that $\gamma_n \mapsto -\gamma_n$ is a state-preserving $*$-isomorphism we get

$$\left\| XD_n^{\frac{1}{p}} \right\|_p^2 \geq \left\| (a + y_n d)D_n^{\frac{1}{p}} \right\|_p^2 + (p-1) \left\| (\gamma_n b + \gamma_n^* c)D_n^{\frac{1}{p}} \right\|_p^2 = I + (p-1)II.$$

Key estimates 2

- The estimate of I and II heavily depends on the structure of the algebra generated by Γ_{n-1} and y_n. Indeed, we have the following state-preserving $*$-isomorphism.

$$\Phi : (\Gamma_{\langle 1,\cdots,n-1,y_n\rangle}, \tau^{\varepsilon}) \to (\ell_2^{\infty}(\Gamma_{\langle 1,\cdots,n-1\rangle}), \psi \otimes \tau^{\varepsilon}) \subseteq M_2 \otimes \Gamma_{\langle 1,\cdots,n-1\rangle},$$

$$a + y_n d \quad \mapsto \quad \begin{bmatrix} a + \mu_n^2 d & 0 \\ 0 & a - \mu_n^{-2} d \end{bmatrix},$$

where ψ is a tracial state on ℓ_2^{∞} given by

$$\psi(x, y) = \lambda x + (1-\lambda) y, \quad x, y \in \mathbb{C}$$

and $\lambda = \frac{1}{1+\mu^4}$.

- An asymmetric convexity inequality as follows.

$$\left(\lambda \left\|A + \mu^2 B\right\|_p^p + (1-\lambda) \left\|A - \mu^{-2} B\right\|_p^p\right)^{\frac{2}{p}} \geq \|A\|_p^2 + \frac{1}{3\mu^4}(p-1) \|B\|_p^2 .$$

Approximation by the central limit procedure

- For the increased index set

$$\widetilde{I} = \{(i,j) : 1 \leq i \leq n,\ 1 \leq j \leq m\} \cup \{(-i,-j) : 1 \leq i \leq n,\ 1 \leq j \leq m\},$$

  we construct generalized baby gaussians $\gamma_{i,j}$ associated with the parameter μ_i for $1 \leq i \leq n$, $1 \leq j \leq m$.

- Note that the "choice of sign" function ε in this case would be

$$\varepsilon : \widetilde{I} \times \widetilde{I} \to \{\pm 1\}$$

  satisfying a similar conditions.

- Now we replace $\varepsilon((i_1, i_2), (j_1, j_2))$, $(i_1, i_2) \prec (j_1, j_2) \in \widetilde{I}$ with a family of i.i.d. random variables with

$$P(\varepsilon((i_1, i_2), (j_1, j_2)) = -1) = \frac{1-q}{2}, \quad P(\varepsilon((i_1, i_2), (j_1, j_2)) = 1) = \frac{1+q}{2},$$

  where $(i_1, i_2) \prec (j_1, j_2)$ means $i_1 < j_1$ or $i_1 = j_1, i_2 < j_2$.

Approximation by the central limit procedure: continued

- We set
$$s_{i,m} = \frac{1}{\sqrt{m}} \sum_{j=1}^{m} \gamma_{i,j}.$$
Then, the Speicher's central limit procedure tells us that:
For any $*$-polynomial Q in n non-commuting variables we have
$$\lim_{m\to\infty} \tau^{\varepsilon}(Q(s_{1,m}, \cdots, s_{n,m})) = \tau_q(Q(g_{q,1}, \cdots, g_{q,n}))$$
for almost every ε.
- Since the set of all non-commuting $*$-polynomials is countable, we can find a choice of sign ε such that the above is true for any Q.

Approximation by the central limit procedure: continued

- We can transfer the above convergence in L_p-setting following A. Nou's ultraproduct approach:
  Let $\mathcal{U}$ be a fixed free ultrafilter on $\mathbb{N}$ and $1 \leq p \leq 2$. For any $*$-polynomial Q in n non-commuting variables we have

$$\lim_{m,\mathcal{U}} \left\| Q(s_{1,m}, \cdots, s_{n,m}) D_m^{\frac{1}{p}} \right\|_p = \left\| Q(g_{q,1}, \cdots, g_{q,n}) D_q^{\frac{1}{p}} \right\|_p .$$

  and with a careful analysis we also have

$$\lim_{m,\mathcal{U}} \left\| P_t^{\varepsilon}(Q(s_{1,m}, \cdots, s_{n,m}) D_m^{\frac{1}{p}} \right\|_p = \left\| P_t^{q}(Q(g_{q,1}, \cdots, g_{q,n}) D_q^{\frac{1}{p}} \right\|_p .$$

- Combining the above approximation and the baby version of hypercontractivity we get the "if" direction.

1-dimensional estimate

- g_i: q-gaussian with parameter μ_i.
  For $1 \leq p < 2$ we have

$$\left\| g_i D_n^{\frac{1}{p}} \right\|_p \sim \mu_i^{2-\frac{4}{p}}$$

  whilst $\left\| g_i D_n^{\frac{1}{2}} \right\|_2 = 1$. Thus, P_t^q can not be extended to a contraction from $L^p(\Gamma_q)$ into $L^2(\Gamma_q)$.
- When $p \to 1$ we have a more precise estimate. Indeed, for $\frac{1}{p} = 1 - \frac{1}{2n}$, $n(\geq 2) \in \mathbb{N}$, $\left\| P_q^t \right\|_{L_p \to L_2} \leq 1$ implies that

$$e^{-2t} \leq 2\alpha_\mu^{4-\frac{8}{p}}(p-1).$$

Final remarks

- Our results are far from the optimal one. Even though we restrict ourselves to the tracial case $\mu \equiv 1$ we do not get the optimal estimate.
- Note that we did not use the Jordan-Wigner transformation! Actually, if we adapt our approach to the tracial case $\mu \equiv 1$, we can prove the tracial optimal hypercontractivity without using the Jordan-Wigner transformation.
- The second quantization procedure in the twisted baby Fock model is not available!