LARGE DEVIATION PRINCIPLE FOR PERIODIC STATES OF QUANTUM SPIN SYSTEMS

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$$I^{\phi}(\omega) = \begin{cases} P(\phi) + \omega(A_{\phi}) - s(\omega) & \text{if } \omega \in \mathscr{S}^{\mathbb{Z}^d}(\mathcal{A}) \\ \\ +\infty & \text{if } \omega \in \mathscr{S}(\mathcal{A}) \backslash \mathscr{S}^{\mathbb{Z}^d}(\mathcal{A}). \end{cases}$$

$$\{I^{\phi} = 0\} = \mathscr{S}_{\phi}^{\mathbb{Z}^d}(\mathcal{A})$$

$$\forall \overline{G} \cap \mathscr{S}_{\phi}^{\mathbb{Z}^d}(\mathcal{A}) = \emptyset, \qquad \nu_{\alpha}(G) \leq \nu_{\alpha}(\overline{G}) \leq e^{-1/t_{\alpha} \inf_{\overline{G}} \{I^{\phi} - \varepsilon\}}$$

In particular, when $\mathscr{S}_{\phi}^{\mathbb{Z}^d}(\mathcal{A}) = \{\omega_{\phi}\}$ we have

$$\nu_{\alpha} \to \delta_{\omega_{\phi}}$$

For each convex open set $\mathcal{G}$ containing an invariant state we have

$$\lim t_{\alpha} \log \nu_{\alpha}(\mathcal{G}) = \lim t_{\alpha} \log \nu_{\alpha}(\overline{\mathcal{G}}) = -\inf_{\overline{\mathcal{G}}} I^{\phi} = -\inf_{\mathcal{G}} I^{\phi} = -\inf_{\mathcal{G} \cap \mathscr{S}'} I^{\phi},$$

(1) $\lim t_\alpha \log \nu_\alpha(e^{\widehat{g}/t_\alpha}) =: L(\widehat{g}) = P(f+g) - P(f)$

(2) For each $\mu \in \mathcal{M}^\tau(\Omega)$ there exists a sequence (μ_i) such that:

- $\{\mu_i\} = \mathcal{M}^\tau_{g_i}(\Omega)$
- $\mu_i \to \mu$
- $h(\mu_i) \to h(\mu)$

When both above conditions hold we get the LDP with rate function

$$I^f(\mu) = \begin{cases} P(f) - \mu(f) - h(\mu) & \text{if } \mu \in \mathcal{M}^\tau(\Omega) \\ +\infty & \text{if } \mu \in \mathcal{M}(\Omega) \setminus \mathcal{M}^\tau(\Omega). \end{cases}$$

Example: $\Omega = S^{\mathbb{Z}^d}$, $\tau \equiv \text{shift}$, $f \in C(\Omega)$,

$$\nu_{a,f} = \sum_{\xi \in \mathrm{Per}_a} \frac{e^{\sum_{x\in\Lambda(a)} f(\tau^x \xi)}}{\sum_{\xi' \in \mathrm{Per}_a} e^{\sum_{x\in\Lambda(a)} f(\tau^x \xi')}} \, \delta_{\frac{1}{|\Lambda(a)|}\sum_{x\in\Lambda(a)} \delta_{\tau^x \xi}}$$

$$\left(L(\hat{g}) = \lim \frac{1}{|\Lambda(a)|} \log \sum_{\xi\in\mathrm{Per}_a} e^{\sum_{x\in\Lambda(a)} f+g(\tau^x\xi)} - \lim \frac{1}{|\Lambda(a)|} \log \sum_{\xi'\in\mathrm{Per}_a} e^{\sum_{x\in\Lambda(a)} f(\tau^x\xi')} = P(f+g) - P(f) \right)$$

When $f = f_\phi$ one can take

$$\nu_{a,f_\phi} = \sum_{\xi \in \mathrm{Per}_a} \frac{e^{-U_\phi(\Lambda(a))(\xi)}}{\sum_{\xi' \in \mathrm{Per}_a} e^{-U_\phi(\Lambda(a))(\xi')}} \, \delta_{\frac{1}{|\Lambda(a)|}\sum_{x\in\Lambda(a)} \delta_{\tau^x \xi}}$$

because

$$\lim \frac{1}{|\Lambda(a)|} \sup_{\xi\in\mathrm{Per}_a} \left| \sum_{x\in\Lambda(a)} f(\tau^x\xi) + U_\phi(\Lambda(a))(\xi) \right| = 0$$

$$P(\phi) := \lim \frac{1}{|\Lambda(a)|} \log \mathrm{Tr}\ e^{-H_\phi(\Lambda(a))} \overset{\text{variational principle}}{=} \sup_{\omega \in \mathscr{S}^{\mathbb{Z}^d}(\mathcal{A})} \{s(\omega) - \omega(A_\phi)\}$$

Let $\mathcal{B}$ denote the Banach space of relatively short range translation-invariant interactions:

$$\phi \in \mathcal{B} \quad \text{if} \quad \|\phi\| := \frac{1}{|\Lambda|} \sum_{\Lambda \ni 0} ||\phi(\Lambda)|| < +\infty$$

$$\forall \omega \in \mathscr{S}(\mathcal{A}), \qquad V_a(\omega) = \frac{1}{|\Lambda(a)|} \sum_{x \in \Lambda(a)} \omega \circ \tau^x \quad \text{(average)}$$

$$\omega \in \mathscr{S}(\mathcal{A}) \quad \text{is } a\text{-periodic if} \quad \omega = \omega \circ \tau^x \quad \forall x \in \mathbb{Z}^d(a)$$

$$\forall \omega \in \mathscr{S}^{\mathbb{Z}^d}(\mathcal{A}), \qquad U_a(\omega) \quad (a\text{-periodization})$$

$$\forall A \text{ local self-adjoint}, \qquad \lim \sup_{\omega \in \mathscr{S}^{\mathbb{Z}^d}(\mathcal{A})} |V_a \circ U_a(\omega)(A) - \omega(A)| = 0$$

Theorem 1. *Let* Per_a *be a finite set of* a*-periodic states for all* $a \in \mathbb{Z}^d_>$*, and for each* $\phi \in \mathcal{B}$ *we endow* Per_a *with the probability measure*

$$p_{\phi,\mathrm{Per}_a} = \sum_{\omega \in \mathrm{Per}_a} \frac{e^{s_{\Lambda(a)}(\omega) - \omega(H_\phi(\Lambda(a)))}}{\sum_{\omega' \in \mathrm{Per}_a} e^{s_{\Lambda(a)}(\omega') - \omega'(H_\phi(\Lambda(a)))}} \, \delta_\omega.$$

We assume that the limit

$$l = \lim |\Lambda(a)|^{-1} \log \mathrm{Card}(\mathrm{Per}_a)$$

exists and is finite.

a) The following statements are equivalent:

(i) *For each* $\phi \in \mathcal{B}$ *the net* $(\sum_{\omega \in \mathrm{Per}_a} p_{\phi,\mathrm{Per}_a}(\omega) \delta_{V_a(\omega)})$ *satisfies a large deviation principle in* $\mathscr{S}(\mathcal{A})$ *with powers* $(|\Lambda(a)|^{-1})$ *and rate function*

$$I^\phi(\omega) = \begin{cases} P(\phi) + \omega(A_\phi) - s(\omega) & \text{if } \omega \in \mathscr{S}^{\mathbb{Z}^d}(\mathcal{A}) \\ \\ +\infty & \text{if } \omega \in \mathscr{S}(\mathcal{A}) \backslash \mathscr{S}^{\mathbb{Z}^d}(\mathcal{A}). \end{cases}$$

(ii) *For each* $\phi \in \mathcal{B}$ *we have*

$$l = \lim \varepsilon_{\to 0} \liminf_a |\Lambda(a)|^{-1} \log \mathrm{Card}(\{\omega \in \mathrm{Per}_a : I^\phi(V_a(\omega)) \leq \varepsilon\}).$$

(iii) *For each* $\phi \in \mathcal{B}$ *we have*

$$\lim \frac{1}{|\Lambda(a)|} \log \sum_{\omega \in \mathrm{Per}_a} e^{s_{\Lambda(a)}(\omega) - \omega(H_\phi(\Lambda(a))} = l + P(A_\phi).$$

The above equivalences hold verbatim replacing for each $a \in \mathbb{Z}^d_>$ *and each* $\omega \in \mathrm{Per}_a$ *the quantity* $s_{\Lambda(a)}(\omega)$ *in the definition of* p_{ϕ,Per_a} *by any real* $s_a(\omega)$ *fulfilling*

$$|\Lambda(a)| s(V_a(\omega)) \leq s_a(\omega) \leq s_{\Lambda(a)}(\omega).$$

b) Let $D = \{\varphi_n : n \in \mathbb{N}\}$ *be a countable dense set in* $\mathcal{B}$ *and let* $\omega_{\varphi_n} \in \mathscr{S}^{\mathbb{Z}^d}_{\varphi_n}(\mathcal{A})$ *for all* $n \in \mathbb{N}$*. Then all the conditions of part a) hold if* $l = 0$ *and for each* $n \in \mathbb{N}$ *we have eventually*

$$\mathrm{Per}_a \supset \{U_a(\omega_{\varphi_k}) : 1 \leq k \leq n\}.$$

Example 2. Take $\mathrm{Per}_a = \{U_a(\omega_{\varphi_k}) : 1 \leq k \leq |a|\}$, or more generally $\mathrm{Per}_a = \{U_a(\omega_{\varphi_k}) : 1 \leq k \leq n_a\}$ with $\lim |\Lambda(a)|^{-1} \log n_a = 0$.

Corollary 3. *Each limit point of the net* $\left(\sum_{\omega\in\mathrm{Per}_a} p_{\phi,\mathrm{Per}_a}(\omega)\ V_a(\omega)\right)$ *is an equilibrium state for* ϕ*; in particular* $\lim\sum_{\omega\in\mathrm{Per}_a} p_{a,\phi}(\omega)\ V_a(\omega) = \omega_\phi$ *when* ω_ϕ *is the unique equilibrium state for* ϕ*.*

Corollary 4. *For each convex open set* $\mathscr{G}$ *containing some invariant state we have*

$$\lim \max_{\omega\in\mathrm{Per}_a, V_a(\omega)\in\mathscr{G}} \left\{s_{\Lambda(a)}(\omega) - \omega(H_{\phi_a}(\Lambda(a)))\right\} = \sup_{\omega\in\mathscr{G}\cap\mathscr{S}^{\mathbb{Z}^d}(\mathcal{A})} \{s(\omega) - \omega(A_\phi)\}$$
$$= \sup_{\omega\in\overline{\mathscr{G}}\cap\mathscr{S}^{\mathbb{Z}^d}(\mathcal{A})} \{s(\omega) - \omega(A_\phi)\}$$

and we can replace $\mathscr{G}$ *by* $\overline{\mathscr{G}}$ *in the left-hand side of both equalities. For each* $\omega\in\mathscr{S}^{\mathbb{Z}^d}(\mathcal{A})$ *and each convex local basis* $\mathscr{G}_\omega$ *at* ω *we have*

$$s(\omega) = \omega(A_\phi) + \inf_{\mathscr{G}\in\mathscr{G}_\omega} \max_{\omega\in\mathrm{Per}_a, V_a(\omega)\in\mathscr{G}} \left\{\frac{1}{|\Lambda(a)|}\left(s_{\Lambda(a)}(\omega) - \omega(H_{\phi_a}(\Lambda(a)))\right)\right\}.$$

Corollary 5. *Let* D *be a dense subset of* $\mathcal{B}$*, let* ω_φ *be an equilibrium state for* φ *for all* $\varphi\in D$*, and let* $a\to\infty$*. Then for each invariant state* ω*, there is a sequence* (φ_n) *in* D*, and a subsequence* (a_{α_n}) *satisfying the following properties:*

(i) $\lim V_{a_{\alpha_n}}\circ U_{a_{\alpha_n}}(\omega_{\varphi_n}) = \omega$*;*

(ii) $s(\omega)\le s(V_{a_{\alpha_n}}\circ U_{a_{\alpha_n}}(\omega_{\varphi_n})) \le \frac{s_{\Lambda(a_{\alpha_n})}}{|\Lambda(a_{\alpha_n})|}(U_{a_{\alpha_n}}(\omega_{\varphi_n}))$ *for all* $n\in\mathbb{N}$*;*

(iii) *The sequences* $(s(V_{a_{\alpha_n}}\circ U_{a_{\alpha_n}}(\omega_{\varphi_n})))$ *and* $(\frac{s_{\Lambda(a_{\alpha_n})}}{|\Lambda(a_{\alpha_n})|}(U_{a_{\alpha_n}}(\omega_{\varphi_n})))$ *are decreasing and converge to* $s(\omega)$*.*

Moreover, (i)-(iii) hold verbatim replacing $V_{a_{\alpha_n}}\circ U_{a_{\alpha_n}}(\omega_{\varphi_n})$ *by* ω_{φ_n}*.*