Dilating completely positive semigroups by coactions of quantum groups

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(φ, ψ)-Markov maps

Definition

(M, φ), (N, ψ) two von Neumann algebras equipped with normal faithful states. A linear map $T : M \to N$ is called a (φ, ψ)-*Markov map* if

1. T is completely positive,
2. $T(\mathbf{1}_M) = \mathbf{1}_N$,
3. $\psi \circ T = \varphi$,
4. $T \circ \sigma_t^{\varphi} = \sigma_t^{\psi} \circ T$.

A (φ, ψ)-Markov map $T : (M, \varphi) \to (N, \psi)$ has an *adjoint* (ψ, φ)-Markov map $T^* : (N, \psi) \to (M, \varphi)$ uniquely determined by

$$\psi\big(T(x)y\big) = \varphi\big(xT(y)\big) \quad \forall x \in M, \forall y \in N.$$

If $(N, \psi) = (M, \varphi)$, we call T also a φ-Markov map.

Factorizable Markov maps

Definition

(Claire Anantharaman-Delaroche, 2004) A (φ, ψ)-Markov map is called *factorizable* if there exists (P, χ) a von Neumann algebra with normal faithfull state and two $*$-endomorphisms

$$i : (M, \varphi) \to (P, \chi) \quad j : (N, \psi) \to (P, \chi)$$

that are also Markov maps such that

$$T = j^* \circ i.$$

Problem

Classify factorizable Markov maps.

Markov dilations

Definition

(Kümmerer, 1980's) A φ-Markov map $T : (M, \varphi) \to (M, \varphi)$ has a *Markov dilation* if there exist

- (N, ψ) a von Neumann algebra with normal faithfull state,
- $i : (M, \varphi) \to (N, \psi)$ a (φ, ψ)-Markov embedding,
- $\alpha : (N, \psi) \to (N, \psi)$ a $*$-automorphism and ψ-Markov map

such that $T^n = i^* \circ \alpha^n \circ i$ for all $n \in \mathbb{N}$

Remark

Can also replace $n \in \mathbb{N}$ by a continuous parameter $t \in \mathbb{R}_+$.

Problem

Which φ-Markov maps can be dilated in this sense?

Theorem

(Kümmerer, 1986) + (Anantharaman-Delaroche, 2004)
(Observed by Köstler, May 2008)
Let $T : (M, \varphi) \to (M, \varphi)$ be a φ-Markov map.
TFAE

1. *T is factorizable.*
2. *T has a Markov dilation (in the sense of Kümmerer).*

Positive results (discrete time)

Theorem

(Ricard, 2008) The class of factorizable Markov maps is stable under composition, adjoints, convex combinations, weak*-limits. Markov maps which are self-adjoint Schur multipliers are factorizable.

From now on: $M = M_n(\mathbb{C})$, $\varphi = \mathrm{tr}$ the normalized trace.
Recall: $T : M_n \to M_n$ a *Schur multiplier* if $T(a_{jk}) = (t_{jk}a_{jk})$ with some $(t_{jk}) \in M_n(\mathbb{C})$.

Negative results (discrete time)

Theorem

(Kümmerer, 1986) (Haagerup+Musat, 2008)

- $\exists$ *tr-Markov maps on* M_n *that are not factorizable for* $n \geq 3$,
- $\exists$ *Schur multipliers on* M_n *that are non-factorizable tr-Markov maps for* $n \geq 4$,
- $\exists$ *Schur multipliers on* M_n *that are factorizable tr-Markov maps, but outside the convex hull of* $\mathrm{Aut}(M_n)$ *for* $n \geq 6$.

Positive results in continuous time

Theorem

(Kümmerer+Maassen, 1987) $(T_t)_{t\geq 0}$ *semigroup of tr-Markov maps, pointwise weak$*$-continuous.*
If $(T_t)_{t\geq 0}$ *is in the convex hull of* $Aut(M_n)$*, then it can be dilated using a Lévy process with values in* $Aut(M_n)$*. In particular, if* $(T_t)_{t\geq 0}$ *is semigroup of tr-Markov maps and* $T_t^* = T_t$ *for all* $t \geq 0$*, then*

$$T_t \in \mathrm{conv}(\mathrm{Aut}(M_n)) \quad \forall t \geq 0$$

Negative results in continuous time

Theorem

(Musat+Haagerup, 2008) For $n \geq 4$ there exists a continuous one-parameter semigroup $T_t = e^{-tL}$, $t \geq 0$ of tr-Markov Schur multipliers on M_n such that T_t is not factorizable for small values of $t > 0$.

A characterisation of factorizable maps

Theorem

(Bhat+Skalski, Haagerup+Musat, 2008) Let $T : M_n \to M_n$ be a tr-Markov map. Then T is factorizable if and only if there exists a finite von Neumann algebra N with f.n. trace τ and a unitary $u \in M_n \otimes N$ s.t.

$$T(x) = (\mathrm{id}_{M_n} \otimes \tau)\big(u^*(x \otimes 1)u\big)$$

for $x \in M_n$.

Equivalent formulation

Let $\mathcal{U}_n$ be the "noncommutative coefficient algebra of the unitary group U_n" (Brown algebra) with generators u_{jk}, $1 \leq j, k \leq d$. This is a C^*-bialgebra, which coacts on M_n by

$$\gamma : M_n \ni x \mapsto u^*(x \otimes 1)u \in M_n \otimes \mathcal{U}_n \cong M_n(\mathcal{U}_n)$$

where $u = \sum e_{jk} \otimes u_{jk} \cong (u_{jk}) \in M_n \otimes \mathcal{U}_n \cong M_n(\mathcal{U}_n)$.
A tr-Markov map $T : M_n \to M_n$ is factorizable if and only if there exists a tracial state $\tilde{\tau}$ on $\mathcal{U}_n$ such that $(M_n \otimes \mathcal{U}_n, \mathrm{tr} \otimes \tilde{\tau})$ and

$$\begin{aligned} \gamma &: M_n \ni x \mapsto u^*(x \otimes 1)u \in M_n \otimes \mathcal{U}_n \cong M_n(\mathcal{U}_n) \\ i &: M_n \ni x \to x \otimes 1 \in M_n \otimes \mathcal{U}_n \end{aligned}$$

give a factorization of T.

Observation

(e.g. Schürmann, 1990) If we give up the preservation of the state, then we can dilate *any* unital CP-map on M_n using a coaction of a quantum (semi-)group $\mathcal{U}_n$ (the non-commutative coefficient algebra of the unitary group). This dilation generalizes the "essentially commutative" dilations by Kümmerer and Maassen, it preserves the state if the states of the underlying quantum Lévy processes on $\mathcal{U}_n$ are traces.

Study Lévy processes on $\mathcal{U}_n$ whose states are traces!