Trotter Product formula for quantum stochastic flows

Debashish Goswami

Statistics and Mathematics Unit
Indian Statistical Institute, Kolkata

goswamid@isical.ac.in

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contents

- **Trotter product formula for semigroups**:
  $(T_t), (S_t)$ be two C_0-contractiion semigroups on a Banach space, and assume that their generators, say A and B respectively, have a dense common domain and $A + B$ is the pre-generator of a C_0 contractive semigroup, say W_t. Then we have the following formula for W_t:

$$W_t(x) = \lim_{n \to \infty} (T_{t/n} S_{t/n})^n (x), \quad x \in X.$$

- Our goal: generalize the above to the framework of quantum stochastic flows.
- This is joint work with K B Sinha and B. Das.

Previous works

- In 1982, Parthasarathy-Sinha obtained a stochastic Trotter Product formula for unitary operator-valued evolutions, constituted from independent increments of classical Brownian motion.
- More recently this was extended by Lindsay and Sinha to the flows constituted from the fundamental quantum processes, satisfying Hudson-Parthasarathy type quantum stochastic differential equations (q.s.d.e for short), however, with only **bounded** operator coefficients.

Definition

Let $\mathcal{A}$ be a C^* or von Neumann algebra, k_0 Hilbert space with orthonormal basis $\{e_i\}$. We say that a family of completely positive contractive (CPC) maps (also normal in case $\mathcal{A}$ is a von-Neumann algebra) $(j_t)_{t \geq 0}$ from a unital C^* or von Neumann algebra $\mathcal{A}$ to $\mathcal{A}'' \otimes B(\Gamma)$ ($\Gamma := \Gamma(L^2(\mathbb{R}_+, k_0))$), is a (quantum stochastic) CPC flow, with noise space k_0 and (possibly unbounded, linear) 'structure maps' $\{\theta^\mu_\nu, \ \mu, \nu \in \{0\} \cup \{1, 2, ...dimk_0\}\}$, if the following holds:
(i) There is a dense $*$-subalgebra $\mathcal{A}_0$ of $\mathcal{A}$ (norm dense for C^* algebra and ultraweakly dense for von-Neumann algebra) such that $\mathcal{A}_0$ is contained in the domain of all the maps θ^μ_ν,
(ii) For $u, v \in \mathcal{H}, \ f, g \in L^2(\mathbb{R}_+, k_0)$ and $x \in \mathcal{A}_0$:

$$< j_t(x)ue(f), ve(g) >=< xue(f), ve(g) >$$
$$+ \sum_{\mu,\nu} \int_o^t < j_s(\theta^\mu_\nu(x))ue(f), ve(g) > g^\mu(s) f_\nu(s) ds.$$

Here, $f^i(s) = \langle e_i, f(s) \rangle$, $f_i(s) = \overline{f^i(s)}$, $f_0(s) = f^0(s) = 1$.

- Symbolically,

$$dj_t(x) = \sum_{\mu,\nu} j_t(\theta^\mu_\nu(x))\Lambda^\nu_\mu(dt), \quad j_0 =]rmid.$$

- $\Lambda^\mu_\nu(s)$ are the fundamental integrators of (Hudson-Parthasarathy) quantum stochastic calculus. satisfying the quantum-Ito formula:

$$d\Lambda^\alpha_\beta(t) d\Lambda^\mu_\nu(t) = \hat{\delta}^\alpha_\nu d\Lambda^\mu_\beta(t)$$

for $\alpha, \beta = 0, 1, 2, 3....$, and

$$\begin{aligned} \hat{\delta}^\alpha_\beta &:= 0 \ if \ \alpha = 0 \ or \ \beta = 0 \\ &:= \delta_{\alpha,\beta} \ otherwise, \end{aligned} \qquad (1)$$

- Structure maps can be written in a copmpact form $(\mathcal{L}, \delta, \sigma)$, or in the matric form:
$$\begin{pmatrix} \mathcal{L} & \delta^\dagger \\ \delta & \sigma \end{pmatrix},$$
where $\sigma := \sum_{i,j} \theta^i_j(x) \otimes |e_j >< e_i|$, $\delta(x) := \sum_i \theta^i_0(x) \otimes e_i$, $\delta^\dagger(x) := \delta(x^*)^*$, and $\mathcal{L}(x) = \theta^0_0(x)$, for $x \in \mathcal{A}_0$.
- Necessary conditions for j_t to be $*$-homomorphism for all t:

$$\theta^\mu_\nu(xy) = \theta^\mu_\nu(x)y + x\theta^\mu_\nu(y) + \sum_{i=1}^{dimk_0} \theta^\mu_i(x)\theta^i_\nu(y), \quad \theta^\mu_\nu(x)^* = \theta^\nu_\mu(x^*). \tag{2}$$

This is equivalent to $\mathcal{L}(x^*) = \mathcal{L}(x)^*$, $\sigma(x) = \pi(x) - x \otimes 1_{k_0}$, where π is $*$-homomorphism, δ being π-derivation, and the cocycle relation $\mathcal{L}(x^*y) - \mathcal{L}(x^*)y - x^*\mathcal{L}(y) = \delta(x)^*\delta(y)$.

Definition

The time shift operator θ_t, $\theta_t : L^2(\mathbb{R}_+) \to L^2([t,\infty))$ *is defined as*

$$\begin{aligned} \theta_t(f)(s) &= 0 && \text{if } s < t \\ &= f(s-t) && \text{if } s \geq t\ . \end{aligned} \tag{3}$$

Let $\Gamma(\theta_t)$ denotes its second quantization, that is $\Gamma(\theta_t)(e(g)) = e(\theta_t(g))$, for g in $L^2(\mathbb{R}_+, k_0)$ and extended linearly as an isometry on whole $\Gamma(L^2(\mathbb{R}_+, k_0))$. For $X \in \mathcal{A} \otimes B(\Gamma_{[r,s]})$,

$$\Gamma(\theta_t)(X \otimes I_{\Gamma^s})\Gamma(\theta_t^*) \;=\; P_{12}(|\Omega_t >< \Omega_t| \otimes 1_{\Gamma^t_{r+t}} \otimes \hat{X} \otimes I_{\Gamma^{t+s}})P_{12}^*,$$

where $P_{12} : \Gamma_t \otimes h \otimes \Gamma^t \longrightarrow h \otimes \Gamma_t \otimes \Gamma^t (\cong h \otimes \Gamma)$ is the unitary flip between first and second tensor components.

Let $\xi_t : B(h \otimes \Gamma_s^r) \longrightarrow B(h \otimes \Gamma_{t+s}^{t+r})$ be given by :

$$\xi_t(X) = \hat{X}.$$

Definition

A CPC flow j_t is called a cocycle if

$$j_{s+t}(x) = j_s \circ \xi_s \circ j_t(x), \text{ for } x \in \mathcal{A}.$$

Henceforth, all the CPC flows considered are assumed to be cocycles, and we shall refer to them as CPC cocycles..

Lemma

For a CPC cocycle j_t, with structure maps defined on $\mathcal{A}_0$ as considered before, $j_t^{c,d}(x)$ defined by $\langle e(c1_{[0,t]}), j_t(x)e(d1_{[0,t]})\rangle$ is a C_0 semigroup on $\mathcal{A}$. Furthermore the restriction of the generator of $j_t^{c,d}(x)$ to $\mathcal{A}_0$ is
$\mathcal{L} + \langle c, \delta\rangle + \delta_d^\dagger + \langle c, \sigma_d\rangle + \langle c, d\rangle\, id$.

Let $\mathcal{A}$ be a C^* or von-Neumann algebra which is equipped with a faithful, semifinite and lower-semicontinuous trace τ. Suppose we are given two CPC cocycles

$$j_t^{(1)} : \mathcal{A} \longrightarrow \mathcal{A}'' \otimes B(\Gamma(L^2(\mathbb{R}_+, k_1)))$$

and

$$j_t^{(2)} : \mathcal{A} \longrightarrow \mathcal{A}'' \otimes B(\Gamma(L^2(\mathbb{R}_+, k_2))),$$

which structure maps $(\mathcal{L}^{(1)}, \delta^{(1)}, \sigma^{(1)})$ and $(\mathcal{L}^{(2)}, \delta^{(2)}, \sigma^{(2)})$ respectively. *In the following, we assume that the hypothesis in the definition (2.1) is true for both sets of structure maps with the same* $\mathcal{A}_0$. Let $\Gamma_1 := \Gamma(L^2(\mathbb{R}_+, k_1))$ and $\Gamma_2 := \Gamma(L^2(\mathbb{R}_+, k_2))$. For $c^{(j)}, d^{(j)} \in k_j,\ j = 1, 2$, define $j_t^{c^{(j)}, d^{(j)}} = j_t^{(j)\ c^{(j)}, d^{(j)}}$. We now define the Trotter product of these two flows:
For $x \in \mathcal{A}$, define $\eta_t : \mathcal{A} \longrightarrow \mathcal{A} \otimes B(\Gamma_1 \otimes \Gamma_2)$ by :

$$\eta_t(x) = (j_t^{(1)} \otimes id_{B(\Gamma_2)}) \circ j_t^{(2)}(x). \tag{4}$$

Take a dyadic partition of the whole real line $\mathbb{R}$ and consider the part of the partition in $[s,t]$ for large n, described in the picture below:

$$----\Big|_{[2^n s]\cdot 2^{-n}} --- \Big[s -- \Big|_{([2^n s]+1)\cdot 2^{-n}} ---------\Big|_{[2^n t]\cdot 2^{-n}} -$$

where $[t] =$ *integer* $\leq t$ for real t.

Definition

Set

$$\phi^{(n)}_{[s,t]} = [\big(\xi_s \circ \eta_{([2^n s]+1)2^{-n}}\big)] \circ \left\{ \prod_{j=[2^n s]+1}^{[2^n t]-1} \Big(\xi_{j.2^{-n}} \circ \eta_{2^{-n}} \otimes 1_{B\left(\Gamma^{j.2^{-n}}_{(j+1).2^{-n}}\right)} \Big) \right\} \circ [\big(\xi_{[2^n t].2^{-n}} \circ \eta_{t-[2^n t].2^{-n}}\big)]. \quad (5)$$

Set $\phi^{(n)}_t := \phi^{(n)}_{[0,t]}$. *The map* $\phi^{(n)}_t$ *will be called the n-fold Trotter product of the flows* $j^{(1)}_t$ *and* $j^{(2)}_t$.

Using the semigroup Trotter product formula, it is not difficult to prove the following:

Theorem

The (weak) Trotter product formula-I :
Suppose $\mathcal{A}$ is a C^*-algebra and that for each c_j, d_j belonging to $k_j, j = 1, 2$, the closure of the operator $\sum_{j=1}^{2} \left(\mathcal{L}^{(j)} + \langle c_j, \delta^{(j)} \rangle + \delta_{d_j}^{\dagger(j)} + \langle c_j, \sigma_{d_j} \rangle + \langle c_j, d_j \rangle \right)$ *generates a C_0 contractive semigroup in $\mathcal{A}$.*
Then $\phi_t^{(n)}(x)$ as defined above converges in the weak operator topology of $h \otimes \Gamma^1 \otimes \Gamma^2$ to $j_t(x)$ where j_t is another CPC flow satisfying a q.s.d.e. with structure matrix

$$\begin{pmatrix} \mathcal{L}^{(1)} + \mathcal{L}^{(2)} & \delta^{\dagger(1)} & \delta^{\dagger(2)} \\ \delta^{(1)} & \sigma^{(1)} & 0 \\ \delta^{(2)} & 0 & \sigma^{(2)} \end{pmatrix}.$$

Theorem

The (Weak) Trotter product formula-II : *Let $\mathcal{A}$ be a C^* or von-Neumann algebra, and τ be a trace on it. Furthermore assume that:*

(a) *in the structure matrices associated with $j_t^{(1)}$ and $j_t^{(2)}$, $\sigma^{(j)} = 0$ for $j = 1, 2$,*

(b) *the closure of $\mathcal{L}_2^{(1)} + \mathcal{L}_2^{(2)}$ generates a C_0, contractive, analytic semigroup in $L^2(\tau)$.*

Then $\phi_t^{(n)}(x)$ as defined above converges in the weak operator topology of $h \otimes \Gamma^1 \otimes \Gamma^2$ to $j_t(x)$ for all x in $\mathcal{A}$, where j_t is a CPC flow satisfying the q.s.d.e. with structure matrix

$$\begin{pmatrix} \mathcal{L}^{(1)} + \mathcal{L}^{(2)} & \delta^{\dagger(1)} & \delta^{\dagger(2)} \\ \delta^{(1)} & 0 & 0 \\ \delta^{(2)} & 0 & 0 \end{pmatrix}.$$

We now come to the case of $*$-homomorphic cocycles. The theorems 3.1 and 3.2 have established that $\phi_t^{(n)}$ converges weakly to j_t (a CPC cocycle flow) on $h \otimes \Gamma_1 \otimes \Gamma_2 \cong h \otimes \Gamma$. Clearly, when $j_t^{(i)}$ are $*$-homomorphic, each $\phi_t^{(n)}$ is a $*$-homomorphism from $\mathcal{A} \rightarrow \mathcal{A}'' \otimes B(\Gamma)$, and so the above convergence is strong if and only if j_t itself is a $*$-homomorphism. Thus, we can convert the 'Weak Trotter Product Formulae' above to the strong versions if we have techniques to prove $*$-homomorphic property of a cocycle. We now discuss such a result, which is a new (iteration free) method of proving $*$-homomorphic peroperty applicable for a large class of flows with unbounded structure maps. Then we shall return to the Trotter product formula. Note that this new proof of homomorphic peroperty is quite interesting and useful in its own right, and it should enable us to get existence of quantum stochastic dilation (Evans-Hudson type) for new classes of semigroups with unbounded generators.

Assumptions for proving $*$-homomorphic property of QS flow with unbounded strucrure maps

Let $\mathcal{A}$ be a C^* or von-Neumann algebra, equipped with a semifinite, faithful, lower-semicontinuous (also normal in case $\mathcal{A}$ is a von-Neumann algebra) trace τ, and let $\mathcal{A}_0$ be a dense $*$-subalgebra of $\mathcal{A}$ which is also dense in $h(\equiv L^2(\mathcal{A},\tau))$ in the L^2- topology. Assume that $j_t, t \geq 0$ is a CPC flow as above and let $(T_t)_{t\geq 0}$ be given by:
$\langle u, T_t(x)v\rangle = \langle ue(0), j_t(x)ve(0)\rangle \equiv \left\langle u, j_t^{0,0}(x)v\right\rangle$ for $u, v \in h,\ x \in \mathcal{A}$.
Let us first assume the usual necessary algebraic conditions for j_t to be $*$-homomorphic:

1. **A(i)**

$$\theta^\mu_\nu(xy) = \theta^\mu_\nu(x)y + x\theta^\mu_\nu(y) + \sum_{i=1}^{dimk_0} \theta^\mu_i(x)\theta^i_\nu(y), \quad \theta^\mu_\nu(x)^* = \theta^\nu_\mu(x^*). \tag{6}$$

We now make more assumptions, which are analytic in nature:

A(ii) For each $t \geq 0$, T_t extends as a bounded operator (which we again denote by T_t,) on the Hilbert space h such that $(T_t)_{t \geq 0}$ is a L^2-contractive, C_0-semigroup of operators in the Hilbert space h as well as on $\mathcal{A}$ (w.r.t. norm or ultraweak topology depending on C^* or von Neumann case). On h, T_t is further assumed to be an analytic semigroup. We shall denote by $\mathcal{L}_2$ the generator of $((T_t)_{t \geq 0})$ in h.

A(iii) Suppose that $\mathcal{A}_0 \subseteq D(\mathcal{L}) \cap D(\mathcal{L}_2)$, and that T_t leaves $\mathcal{A}_0$ invariant.

A(iv) For $x \in \mathcal{A}_0$, $\mathcal{L}(x^*x) \in \mathcal{A} \cap L^1(\tau)$ and $\tau(\mathcal{L}(x^*x)) \leq 0$ (a kind of weak dissipativity).

A(v) There exists a total subset $\mathcal{W}$ of $L^2(\mathbb{R}_+, k_0)$, such that for f, g in $\mathcal{W}$, $x \in A \cap L^1(\tau)$ and u,v in $L^\infty(\tau) \cap L^2(\tau)$, we have:

$$sup_{\;0 \leq s \leq t} \left| \left\langle u f^{\otimes^m}, j_t(x) v g^{\otimes^n} \right\rangle \right| \leq C(u, v, f, g, m, n, t) \|x\|_1, \quad (7)$$

such that for fixed u,v,f,g,m,n, $C(u, v, f, g, m, n, t) = O(e^{\beta t})$ for some $\beta \geq 0$.

- **A(iii)** implies $\mathcal{A}_0$ is a core for both $\mathcal{L}$ and $\mathcal{L}_2$. Furthermore observe that because of analyticity in **A(ii)**, the real part of the operator $(-2\mathcal{L}_2)$ exists as an operator and by **A(iv)**, it is non-negative.
- If $(T_t)_{t\geq 0}$ is *symmetric* with respect to τ, i.e. $\tau(T_t(x)y) = \tau(xT_t(y))$, then **A(ii)** follows. If we assume furthermore that T_t is conservative i.e. $T_t(I) = I \ \forall t \geq 0$ and **A(iii)** is valid, then **A(iv)** also follows.
- Consider a typical diffusion process in $\mathbb{R}$ whose generator is of the form:
$$\mathcal{L} = \frac{1}{2}\frac{d}{dx}a^2(x)\frac{d}{dx} + b(x)\frac{d}{dx}.$$
The coefficients a and b are assumed to be smooth and a is assumed to be non-vanishing everywhere. By a suitable change of variable this can be made into symmetric w.r.t. a suitable measure on $\mathbb{R}$. On the other hand, standard Poisson process on $\mathbb{Z}_+$ for which the assumptions **A(i)**- **A(v)** hold, cannot be made symmetric even by a change of measure on the underlying function algebra. So, our set-up covers cases beyond symmetric.

Theorem

Under the above assumptions, j_t is $*$-homomorphic for all t.

Corollary

Suppose that the trace τ on the algebra is finite. Assume **A(i)** *through* **A(v)**, *but replace the assumption of analyticity in condition* **A(i)** *by the following: $\mathcal{A}_0 \subseteq D(\mathcal{L}_2) \cap D(\mathcal{L}_2^*)$. Then the conclusion of the above theorem remains valid.*

Corollary

Suppose the CPC flow $(j_t)_{t \geq 0}$ satisfies **A(i)-A(iv)** *and that for $x \in \mathcal{A} \cap L^1(\tau)$,*

$$\|j_t^{c,d}(x)\|_1 \leq exp(tM)\|x\|_1 \tag{8}$$

for c, d in k_0, where M depends only on $\|c\|, \|d\|$. Then the estimate **A(v)** *and hence the conclusion of the above theorem 4.1 holds.*

Corollary

For a CPC flow $(j_t)_{t\geq 0}$ on a type-I von-Neumann algebra with atomic centre, the conditions **A(i)** *through* **A(iv)** *imply* **A(v)** *and hence also imply that j_t is a $*$ homomorphism.*

Proof.

Observe that in a type-I algebra with atomic centre, we have for $x \in L^1(\tau)$,

$$\|x\|_\infty \leq \|x\|_1.$$

As j_t is a contractive flow, we have that for $x \in L^1(\tau)$,

$$\begin{aligned} &\sup_{0\leq s\leq t} \left| \left\langle uf^{\otimes^m}, j_t(x) vg^{\otimes^n} \right\rangle \right| \\ &\leq \|x\|_\infty \|f^{\otimes^m}\| \|g^{\otimes^n}\| \|u\|_2 \|v\|_2 \\ &\leq \|x\|_1 \|f^{\otimes^m}\| \|g^{\otimes^n}\| \|u\|_2 \|v\|_2. \end{aligned} \tag{9}$$

□

Strong Trotter Product Formula

D.Goswami

Theorem

The (strong) Trotter product formula:

Suppose $\mathcal{A}$ is a C^* algebra. Let $j_t^{(1)}$ and $j_t^{(2)}$ be two $*$-homomorphic quantum stochastic flows satisfying the condition of Weak Trotter Product Formula I, and furthermore, there are constants $M_j \equiv M_j(c_j, d_j)$, $j = 1, 2$ such that

(a) $\| {j_t^{(j)}}^{c_j, d_j}(x) \|_1 \leq exp(tM_j) \|x\|_1$, for $x \in \mathcal{A} \cap L^1(\tau)$, $c_j, d_j \in k_j$, $j = 1, 2$;

(b)$\tau(\mathcal{L}^{(j)}(x^* x)) \leq 0$ for $j = 1, 2$;

(c) each of the semigroups generated by $\mathcal{L}^{(1)}$ and $\mathcal{L}^{(2)}$ as well as their Trotter product limit have analytic $L^2(\tau)$ extensions as semigroups.

Then $\phi_t^{(n)}(x)$ as defined above converges in the strong operator topology of $h \otimes \Gamma_1 \otimes \Gamma_2$ to a $*$-homomorphic quantum stochastic flow j_t.

A similar strong analogue of Weak Trotter Product Formula II also holds.

Applications and examples

- **Brownian motion on compact Lie group:** We can construct the Brownian motion X_t on a compact Lie group G as a limit (in probability) of $X_t^{(n)} := \prod_{i=1}^{k} \prod_{l=0}^{[2^n t]} \exp((W^{(i)}_{\frac{[2^n l]+1}{2^n}} - W^{(i)}_{\frac{l}{2^n}})\chi_i) \to X_t$, where $\{\chi_\ell\}_{\ell=1}^{k}$ is a basis for the Lie algebra of G and $W_t^{(\ell)}$ is the standard Brownian motion on $\mathbb{R}$.
- **Random walk in discrete group:** Similarly, a construction of time homogeneous random walk X_t on a discrete, finitely generated group G, with torsion free generators, $\{g_1, g_2, ...g_{2k}\}$ $(g_{k+l} = g_l^{-1})$, is obtained as the following limit in probability

$$X_t^{(n)} := \prod_{l=0}^{[2^n t]} \prod_{i=1}^{k} \mathcal{G}^{(i)}_{\frac{l+1}{2^n}} (\mathcal{G}^{(i)}_{\frac{l}{2^n}})^{-1} \to X_t,$$

where $(N_t^{(i)})_{t \geq 0}, \ i = 1, \ldots, 2k$ are mutually independent Poisoon processes on $\mathbb{N} \cup \{0\}$, with intensity parameter $(\lambda_i)_{i=1}^{2k}$, respectively, $Z_t^{(i)} := N_t^{(i)} - N_t^{(k+i)}$, and $\mathcal{G}_t^{(l)}(\omega) := g_l^{Z_t^{(l)}(\omega)}$.

Technical preparation: projective tensor product of Banach spaces

For two Banach spaces E_1, E_2, the projective tensor product $E_1 \otimes_\gamma E_2$ is the completion of the algebraic tensor product $E_1 \otimes_{\mathrm{alg}} E_2$ under the cross-norm $\| \cdot \|_\gamma$ given by $\|X\|_\gamma = \inf \sum_i \|x_i\| \|y_i\|$, where infimum is taken over all possible expressions of X of the form $X = \sum_{i=1}^n x_i \otimes y_i$. Suppose $T_j \in B(E_j, F_j)$ where E_j, F_j, for $j = 1, 2$ are Banach spaces. Then $T_1 \otimes_{\mathrm{alg}} T_2$ extends to a bounded operator

$$T_1 \otimes_\gamma T_2 : E_1 \otimes_\gamma E_2 \longrightarrow F_1 \otimes_\gamma F_2$$

with bound

$$\|T_1 \otimes_\gamma T_2\| \leq \|T_1\| \|T_2\|.$$

Lemma

Suppose T_t and S_t are two C_0 semigroups of bounded operators on E_1 & E_2 with generators L_1 and L_2 respectively. Then $T_t \otimes_\gamma S_t$ becomes a C_0 semigroup of operators on $E_1 \otimes_\gamma E_2$ whose generator is the closed extension of the operator $L_1 \otimes_{\mathrm{alg}} 1 + 1 \otimes_{\mathrm{alg}} L_2$, defined on $D(L_1) \otimes_{\mathrm{alg}} D(L_2)$ in the space $E_1 \otimes_\gamma E_2$.

Lemma

Let E be a Banach space, and let A and B belong to $Lin(E, E)$ with dense domains $D(A)$ and $D(B)$ respectively. Suppose there is a total set $D \subset D(A) \cap D(B)$ with the properties :

(i) *$A(D)$ is total in E,* **(ii)** *$\|B(x)\| < \|A(x)\|$ for all $x \in D$.*

Then $(A + B)(D)$ is also total in E.

Proof.

If $A(D) \subseteq (A + B)(D)$, then $F \equiv span\{(A + B)(D)\}$ is dense in E. Therefore w.l.g suppose $\overline{F} \neq E$, so $\exists$ non-zero $y_0 = A(x_0)$, $x_0 \in D$, such that $y_0 \notin (A + B)(D)$. Then by Hahn-Banach theorem, $\exists$ $\Lambda \in E^*$, the topological dual of E, such that $\|\Lambda\| = 1$, $|\Lambda(y_0)| = \|y_0\|$ as well as $\Lambda((A + B)(D)) = 0$. Then $\|y_0\| = |\Lambda(A(x_0))|$ and $|\Lambda(A(x_0))| = |\Lambda(B(x_0))|$. But $|\Lambda(B(x_0))| \leq \|B(x_0)\| < \|A(x_0)\| = \|y_0\|$, which is a contradiction. Therefore $\overline{F} = E$. ∎

Sketch of proof of $*$-homomorphic property

Let $\hat{\mathcal{L}} = \overline{\mathcal{L}_2 \otimes_\gamma 1 + 1 \otimes_\gamma \mathcal{L}_2}$, $C = (-2Re(\mathcal{L}_2))^{\frac{1}{2}}$,
$C \otimes_\gamma C := (C \otimes_\gamma 1) \circ (1 \otimes_\gamma C) = (1 \otimes_\gamma C) \circ (C \otimes_\gamma 1)$ in $h \otimes_\gamma h$,
$\mathcal{F} := \mathcal{A}_0 \otimes_{\mathrm{alg}} \mathcal{A}_0$, and $\mathcal{Y} := \{(\lambda - \hat{\mathcal{L}})^{-1}(x \otimes y)|\ x, y \in \mathcal{A}_0\}$.
For x in $\mathcal{A}_0$,

$$\begin{aligned} &\frac{d}{dt}\|T_t(x)\|^2 \\ &= \ < \mathcal{L}_2(T_t(x)), T_t(x) > + < T_t(x), \mathcal{L}_2(T_t(x)) > \\ &= \ -\|C \circ T_t(x)\|^2, \end{aligned}$$

and integratiion by parts gives

$$\|x\|^2 - \lambda \int_0^\infty e^{-\lambda t}\|T_t(x)\|^2 dt = \int_0^\infty e^{-\lambda t}\|C(T_t(x))\|^2 dt \geq 0,$$

and moreover, for nonzero x and $\lambda > 0$, the inequality is strict, because otherwise $\|T_t(x)\| = 0$ for almost all and hence (by strong continuity of T_t) for all $t \geq 0$, contradicting $T_0(x) = x$.

Lemma

$\|(C \otimes_\gamma C)(X)\|_\gamma \leq \|(\lambda - \hat{\mathcal{L}})(X)\|_\gamma$ *for all* X *in* $D(\hat{\mathcal{L}})$ *and we have strict inequality if* X *is in* $\mathcal{Y}$.

Proof.

It follows from the estimate below for $X = \sum_{i=1}^{k} x_i \otimes y_i \in \mathcal{F}$:

$$\int_0^\infty dt\ e^{-\lambda t} \|C \otimes_\gamma C(T_t \otimes_\gamma T_t)(X)\|_\gamma$$

$$= \int_0^\infty dt\ e^{-\lambda t} \|\sum_{i=1}^{k} C(T_t(x_i)) \otimes C(T_t(y_i))\|_\gamma$$

$$\leq \sum_{i=1}^{k} (\int_0^\infty dt\ e^{-\lambda t} \|C(T_t(x_i))\|^2)^{\frac{1}{2}} (\int_0^\infty dt\ e^{-\lambda t} \|C(T_t(y_i))\|^2)^{\frac{1}{2}}$$

$$\leq \sum_{i=1}^{k} \|x_i\| \|y_i\| \ \text{(strict inequality for nonzero X)},$$

The assumption **A(iv)** as well as the algebraic relations **A(i)** give for $x \in \mathcal{A}_0, \epsilon > 0$

$$\|\theta_0^i(x)\|_h^2 \leq \sum_{j=1}^{\infty} \|\theta_0^i(x)\|_h^2 \leq \|C(x)\|_h^2 \leq \|(C+\epsilon)(x)\|_h^2, \qquad (10)$$

so

$$\sum_{i\geq 1} \|\theta_0^i(x)\|\|\theta_0^i(y)\| \leq \{(\sum_{i\geq 1} \|\theta_0^i(x)\|^2)(\sum_{i\geq 1} \|\theta_0^i(y)\|^2)\}^{\frac{1}{2}} \qquad (11)$$
$$\leq \|(C+\epsilon)x\|\|(C+\epsilon)y\| < \infty.$$

Set $B \in \mathrm{Lin}(D(C) \otimes_{\mathrm{alg}} D(C), L^2(\tau) \otimes_\gamma L^2(\tau))$ by $B(x \otimes y) = \sum_{i\geq 1} \theta_0^i(x) \otimes \theta_0^i(y)$, and observe

$$\|B\{(C+\epsilon)^{-1} \otimes_\gamma (C+\epsilon)^{-1}\}(x \otimes y)\|_\gamma \leq \|x \otimes y\|_\gamma. \qquad (12)$$

So $B\{(C+\epsilon)^{-1} \otimes_\gamma (C+\epsilon)^{-1}\}$ extends to a contraction on $h \otimes_\gamma h$, hence $\|B(X)\|_\gamma \leq \|(C+\epsilon) \otimes_\gamma (C+\epsilon)(X)\|_\gamma$ for all $X \in D(C) \otimes_{\text{alg}} D(C)$, and letting $\epsilon \to 0$,

$$\|B(X)\|_\gamma \leq \|(C \otimes_\gamma C)(X)\|_\gamma \tag{13}$$

for all X in $D(C) \otimes_{\text{alg}} D(C)$. Thus, $C \otimes_\gamma C$ extends to $D(\hat{\mathcal{L}})$ and we can also extend B to $D(\hat{\mathcal{L}})$. So we have

$$\|B(X)\| \leq \|(C \otimes_\gamma C)(X)\|_\gamma \leq \|(\lambda - \hat{\mathcal{L}})(X)\|_\gamma \text{ for all } X \in D(\hat{\mathcal{L}}). \tag{14}$$

Now $span\{\mathcal{Y}\} \subseteq D(\hat{\mathcal{L}})$, and in particular for Y in $\mathcal{Y}$,

$$\|B(Y)\|_\gamma \leq \|(C \otimes_\gamma C)(Y)\|_\gamma < \|(\lambda - \hat{\mathcal{L}})(Y)\|_\gamma. \tag{15}$$

Theorem

Under assumptions **A(i)-A(v)**, j_t *is* $*$*-homomorphic.*

For brevity, we adopt Einstein's summation convention in the proof. For f, g in $\mathcal{W}$, using the quantum Ito formula we get:

$$\begin{aligned}
&\langle j_t(x)ue(f), j_t(y)ve(g)\rangle \\
&= \langle xue(f), yve(g)\rangle + \int_0^t ds[\langle j_s(\theta^\mu_\nu(x))ue(f), j_s(y)ve(g)\rangle\, g^\mu(s)f_\nu(s) \\
&\quad + \langle j_s(x)ue(f), j_s(\theta^\mu_\nu(y))ve(g)\rangle\, f_\mu(s)g^\nu(s) \\
&\quad + \left\langle\, j_s(\theta^i_\mu(x))ue(f), j_s(\theta^i_\nu(x))ve(g)\right\rangle\ f_\mu(s)g^\nu(s)].
\end{aligned} \tag{16}$$

For fixed u, v in $\mathcal{A} \cap h$, f, g in $\mathcal{W}$, we define for each $t \geq 0$, $\phi_t : \mathcal{A}_0 \times \mathcal{A}_0 \to \mathbb{C}$ by

$$\phi_t(x, y) := \langle\, j_t(x)ue(f), j_t(y)ve(g)\rangle \; - \langle\, j_t(y^*x)ue(f), ve(g)\rangle \,. \quad (17)$$

Define for m,n in $\mathbb{N} \cup 0$,

$$\phi_t^{m,n}(x, y) := \frac{1}{(m!n!)^{\frac{1}{2}}}[\left\langle\, j_t(x)uf^{\otimes^m}, j_t(y)vg^{\otimes^n}\right\rangle \; - \left\langle\, j_t(y^*x)uf^{\otimes^m}, vg^{\otimes^n}\right\rangle$$

$$= \frac{1}{m!n!}\frac{\partial^m}{\partial\rho^m}\frac{\partial^n}{\partial\eta^n}\{\langle j_t(x)ue(\rho f), j_t(y)ve(\eta g)\rangle - \langle j_t(y^*x)ue(\rho f), ve(\eta g)\rangle\} \quad (18)$$

From this, we get a recursive integral relation amongst $\phi_t^{m,n}(x, y)$ as follows:

$$\phi_t^{m,n}(x,y) = \int_0^t ds[\phi_s^{m,n}(\theta_0^0(x),y) + \phi_s^{m,n}(x,\theta_0^0(y)) + \phi_s^{m,n}(\theta_0^i(x),\theta_0^i(y))$$
$$+ g^i(s)\phi_s^{m,n-1}(\theta_0^i(x),y) + g^i(s)\phi_s^{m,n-1}(x,\theta_i^0(y))$$
$$+ f_i(s)\phi_s^{m-1,n}(\theta_i^0(x),y) + f_i(s)\phi_s^{m-1,n}(x,\theta_0^i(y))$$
$$+ g^i(s)f_j(s)\phi_s^{m-1,n-1}(\theta_j^i(x),y) + g^i(s)f_j(s)\phi_s^{m-1,n-1}(x,\theta_i^j(y))$$
$$+ g^i(s)\phi_s^{m,n-1}(\theta_0^k(x),\theta_i^k(y)) + f_i(s)\phi_s^{m-1,n}(\theta_i^k(x),\theta_0^k(y))$$
$$+ f_j(s)g^i(s)\phi_s^{m-1,n-1}(\theta_j^k(x),\theta_i^k(y))] \tag{19}$$

where $\phi_t^{-1,n}(x,y) := \phi_t^{m,-1}(x,y) := 0$ for all m, n and x, y.

We set in (19), $m = n = 0$ to get

$$\phi_t^{0,0}(x,y) = \int_0^t ds\{\phi_s^{0,0}(\theta_0^0(x),y) + \phi_s^{0,0}(x,\theta_0^0(y)) + \phi_s^{0,0}(\theta_0^i(x),\theta_i^0(y))\} \tag{20}$$

and if we can show that the hypothesis of this theorem and (20) imply that $\phi_t^{0,0}(x,y) = 0$, then we can embark on our induction hypothesis as

$$\phi_t^{k,l}(x,y) = 0 \text{ for } k + l \leq m + n - 1.$$

Under the induction hypothesis, (19) reduces to

$$\phi_t^{m,n}(x,y) = \int_0^t ds[\phi_s^{m,n}(\theta_0^0(x),y) + \phi_s^{m,n}(x,\theta_0^0(y)) + \phi_s^{m,n}(\theta_0^i(x),\theta_0^i(y))] \tag{21}$$

for $x, y \in \mathcal{A}_0$, which is an equation similar to (20) leading to $\phi_t^{m,n}(x,y) = 0$, as earlier and this will complete the induction process. Thus it only remains to show that the assumptions of this theorem lead to a trivial solution of equation of the type (20).

Omitting the indices m,n, define a map ψ_t belonging to $Lin(\mathcal{A}_0 \otimes_{\rm alg} \mathcal{A}_0, \mathbb{C})$ by:

$$\psi_t(x \otimes y) = \phi_t^{m,n}(x,y),$$

and extend linearly. We have

$$\psi_t(X) = \int_0^t ds[\psi_s((\theta_0^0 \otimes 1 + 1 \otimes \theta_0^0 + \sum_i (\theta_0^i \otimes_{\rm alg} \theta_0^i))(X))], \text{ for X in } \mathcal{F}. \tag{22}$$

The complete positivity of the map j_t implies that

$$\langle j_t(x)\xi, j_t(x)\xi \rangle \leq \langle j_t(x^*x)\xi, \xi \rangle \tag{23}$$

for $\xi \in h \otimes \Gamma$ and hence by **A(v)**, we get that

$$| \left\langle j_t(x)uf^{\otimes^m}, j_t(y)vg^{\otimes^n} \right\rangle | \leq (| \left\langle j_t(x^*x)uf^{\otimes^m}, uf^{\otimes^m} \right\rangle \left\langle j_t(y^*y)vg^{\otimes^n}, vg^{\otimes^n} \right\rangle |$$
$$= O(e^{\beta t}) \|x\|_2 \|y\|_2. \tag{24}$$

The assumptions **A(v)**, Cauchy-Schwartz inequality and (24) together yields

$$|\psi_t(X)| \leq O(e^{\beta t})\|X\|_\gamma, \text{ for } X \in \mathcal{F}, \tag{25}$$

which proves (by denseness of $\mathcal{F}$ in $h \otimes_\gamma h$) that ψ_t extends as a bounded map from $h \otimes_\gamma h$ to $\mathbb{C}$. If we let $G = \hat{\mathcal{L}} + B$, then for $X \in \mathcal{F}$, the equation (22) becomes: $\psi_t(X) = \int_0^t \psi_s(G(X))ds$. By an integration by parts one gets

$$\int_0^\infty dte^{-\lambda t}\psi_t((G-\lambda)(X)) = 0, \text{ for } X \in \mathcal{F}. \tag{26}$$

Using that $\mathcal{F}$ is a core for $\hat{\mathcal{L}}$ and so for G (by (15)) we get the above for all $X \in span\{\mathcal{Y}\}$.. With A in Lemma 5.2 to be $(\hat{\mathcal{L}} - \lambda)$, $D = \mathcal{Y}$, and because of the inequality (15), Lemma 5.2 applies and the denseness of $(G-\lambda)(span\{\mathcal{Y}\})$ follows. Therefore the last equation and (25) lead to

$$\int_0^\infty dte^{-\lambda t}\psi_t(X) = 0 \text{ for all } X \in h \otimes_\gamma h, \text{ for } \lambda > \beta.$$