Noncommutative independence from characters of the infinite symmetric group $\mathbb{S}_\infty$

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(joint with Rolf Gohm)

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Exchangeability in classical probability

The random variables $(X_n)_{n\geq 0}$ are said to be **exchangeable** if

$$\mathbb{E}\big(X_{\mathbf{i}(1)}\cdots X_{\mathbf{i}(n)}\big) = \mathbb{E}\big(X_{\sigma(\mathbf{i}(1))}\cdots X_{\sigma(\mathbf{i}(n))}\big) \qquad (\sigma \in \mathbb{S}_\infty)$$

for all n-tuples $\mathbf{i}\colon \{1,2,\ldots,n\} \to \mathbb{N}_0$ and $n \in \mathbb{N}$.

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Theorem (De Finetti 1931,...)

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"Any exchangeable process is an average of i.i.d. processes."

Exchangeability in noncommutative probability

Given the tracial W*-algebraic probability space $(\mathcal{A}, \varphi)$ the selfadjoint operators $(x_n)_{n \geq 0} \subset \mathcal{A}$ are **exchangeable** if

$$\varphi\big(x_{\mathbf{i}(1)} \cdots x_{\mathbf{i}(n)}\big) = \varphi\big(x_{\sigma(\mathbf{i}(1))} \cdots x_{\sigma(\mathbf{i}(n))}\big) \qquad (\sigma \in \mathbb{S}_\infty)$$

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Theorem (K 2009)

An exchangeable sequence $(x_n)_{n \geq 0} \subset (\mathcal{A}, \varphi)$ is $\mathcal{T}$-**independent**, where

$$\mathcal{T} = \bigcap_{n \geq 0} \mathrm{vN}(x_n, x_{n+1}, x_{n+2}, \ldots)$$

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"What is this noncommutative notion of $\mathcal{T}$-independence?!"

Noncommutative conditional independence

Definition

Let $\mathcal{N}, (\mathcal{M}_i)_{i \in I}$ be von Neumann subalgebras of $(\mathcal{A}, \varphi)$.

NOTATION: $E_{\mathcal{N}}$ is the φ-preserving cond. expectation from $\mathcal{M}$ onto $\mathcal{N}$.

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$$E_{\mathcal{N}}(xy) = E_{\mathcal{N}}(x)E_{\mathcal{N}}(y)$$

for all $x \in \mathrm{vN}(\mathcal{N}, \mathcal{M}_j | j \in J)$ and $y \in \mathrm{vN}(\mathcal{N}, \mathcal{M}_k | k \in K)$, and disjoint subsets J, K of I.

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Equivalent formulation for index set $I = \{1, 2\}$:

$$\begin{array}{ccc} \mathrm{vN}(\mathcal{N}, \mathcal{M}_2) & \subset & \mathcal{M} \\ \cup & & \cup \\ \mathcal{N} & \subset & \mathrm{vN}(\mathcal{N}, \mathcal{M}_1) \end{array}$$

is a **commuting square** (w.r.t. φ).

Exchangeability for the infinite symmetric group $\mathbb{S}_\infty$

$\mathbb{S}_\infty$ is the inductive limit of the symmetric group $\mathbb{S}_n$ as $n \to \infty$, acting on $\{0, 1, 2, \ldots\}$. A positive definite function $\chi \colon \mathbb{S}_\infty \to \mathbb{C}$ is a **character** if it is constant on conjugacy classes and normalized at the identity.

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Elementary observation

Let $\gamma_i := (0, i)$. Then the sequence $(\gamma_i)_{i\in\mathbb{N}}$ is **exchangeable**, i.e.

$$\chi\big(\gamma_{\mathbf{i}(1)}\gamma_{\mathbf{i}(2)}\cdots\gamma_{\mathbf{i}(n)}\big) = \chi\big(\gamma_{\sigma(\mathbf{i}(1))}\gamma_{\sigma(\mathbf{i}(2))}\cdots\gamma_{\sigma(\mathbf{i}(n))}\big)$$

for $\sigma \in \mathbb{S}_\infty$ with $\sigma(0) = 0$, n-tuples $\mathbf{i}\colon \{1, \ldots, n\} \to \mathbb{N}$ and $n \in \mathbb{N}$.

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Task

Identify the convex combination of extremal characters of $\mathbb{S}_\infty$. In other words: **prove a noncommutative de Finetti theorem!**

Thoma's theorem (1964) is a quantum de Finetti theorem!

An extremal character of the group $\mathbb{S}_\infty$ is of the form

$$\chi(\sigma) = \prod_{k=2}^{\infty} \left(\sum_{i=1}^{\infty} a_i^k + (-1)^{k-1} \sum_{j=1}^{\infty} b_j^k \right)^{m_k(\sigma)} .$$

Here $m_k(\sigma)$ is the number of k-cycles in the permutation σ and the two sequences $(a_i)_{i=1}^{\infty}, (b_j)_{j=1}^{\infty}$ satisfy

$$a_1 \geq a_2 \geq \cdots \geq 0, \qquad b_1 \geq b_2 \geq \cdots \geq 0, \qquad \sum_{i=1}^{\infty} a_i + \sum_{j=1}^{\infty} b_j \leq 1.$$

Alternative proofs

Vershik & Kerov 1981: asymptotic representation theory
Okounkov 1997: Olshanski semigroups and spectral theory

A new operator algebraic proof from exchangeability

R. Gohm & C. Köstler. *Noncommutative independence from characters of the symmetric group* $\mathbb{S}_\infty$. 47 pages. Preprint (2010). (`arXiv:1005.5726`)

A helpful reformulation of exchangeability

Theorem (Gohm & K

Suppose the tracial probability space $(\mathcal{A}, \varphi)$ is generated by the sequence $(x_n)_{n \geq 0}$. TFAE:

(a) (x_n) is exchangeable

Remark

Above characterization generalizes easily to sequences of algebras.

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  (i) $x_n = \rho(\sigma_n \sigma_{n-1} \cdots \sigma_1) x_0$ for $n \geq 1$.

NOTATION:
σ_i is the Coxeter generator $(i-1, i)$ of $\mathbb{S}_\infty$, where $\mathbb{S}_\infty$ acts on $\{0, 1, 2, 3, ...\}$ by permutations.

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σ_i is the Coxeter generator $(i-1, i)$ of $\mathbb{S}_\infty$, where $\mathbb{S}_\infty$ acts on $\{0, 1, 2, 3, ...\}$ by permutations. Let $\mathbb{S}_{n,\infty} = \langle \sigma_n, \sigma_{n+1}, \ldots \rangle$.

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Identification of fixed point algebras for unitary representations of $\mathbb{S}_\infty$

Suppose the tracial probability space $(\mathcal{A}, \mathrm{tr})$ is equipped with the (unitary) representation

$$\pi\colon \mathbb{S}_\infty \to \mathcal{U}(\mathcal{A}), \qquad \text{such that } \mathcal{A} = \mathrm{vN}_\pi(\mathbb{S}_\infty).$$

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As before, $\rho := \mathrm{Ad}\,\pi$ is generating with fixed point algebras

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Moreover: $\mathcal{A}_{-1} = \mathrm{vN}(C_k \mid k \in \mathbb{N})$, where $C_k := E_{-1}\big(A_0^{k-1}\big)$,

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Theorem (Gohm & K '09)

$$\mathcal{A}_{-1} = \mathcal{Z}(\mathcal{A}) \qquad\qquad \mathcal{A}_n = \mathcal{A}_0 \vee \mathrm{vN}_\pi(\mathbb{S}_{n+1})$$

Moreover: $\mathcal{A}_{-1} = \mathrm{vN}(C_k \mid k \in \mathbb{N})$, where $C_k := E_{-1}\big(A_0^{k-1}\big)$,
$\mathcal{A}_0 = \mathrm{vN}(A_0, C_k \mid k \in \mathbb{N})$, where $A_0 := E_0\big(\pi(0,1)\big)$.

NOTATION: E_n is the tr-preserving conditional expectation from $\mathcal{A}$ onto $\mathcal{A}_n$.

Cycles

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Let $k \geq 2$. A k-cycle $\sigma = (n_1, n_2, n_3, \ldots, n_k) \in \mathbb{S}_\infty$ is of the form

$$\sigma = \gamma_{n_1}\gamma_{n_2}\gamma_{n_3}\cdots\gamma_{n_{k-1}}\gamma_{n_k}\gamma_{n_1},$$

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Corollary

Disjoint cycles are supported by disjoint sets of star generators.

Cycles & Independence

Theorem (Gohm & K)

Let I, J be subsets of $\mathbb{N}_0$. Then $\mathrm{vN}_\pi(\gamma_i \mid i \in I)$ and $\mathrm{vN}_\pi(\gamma_j \mid j \in J)$ are $\mathcal{A}_0$-independent whenever $I \cap J = \emptyset$.

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Notation

Let $\pi\colon \mathbb{S}_\infty \to \mathcal{U}(\mathcal{A})$ be a (unitary) representation as before. Put

$$v_i := \pi(\gamma_i).$$

Let E_n denote the tr-preserving conditional expectation from $\mathcal{A} = \mathrm{vN}_\pi(\mathbb{S}_\infty)$ onto the fixed point algebra $\mathcal{A}_n$.

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- Every k-cycle is a limit k-cycle for n sufficiently large.
- Limit k-cycles are certain mean ergodic averages of k-cycles. (Compare 'random cycles' in Okounkov's thesis.)
- Limit cycles generate a monoid similar to Olshanski semigroups.

Examples of limit cycles

Lemma (One-shifted representation $n = 1$)

$$E_0(v_{n_1} v_{n_2} v_{n_3} \cdots v_{n_k} v_{n_1}) = \begin{cases} E_0(v_1)^{k-1} & \text{if } n_1 = 0 \\ E_{-1}\Big(E_0(v_1)^{k-1}\Big) & \text{if } n_1 \neq 0 \end{cases}$$

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Corollary (Zero-shifted representation $n = 0$)

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Key observation

Distinguished roles are played by the limit 2-cycle

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(ii) A_0 is trivial $\Leftrightarrow$ $\begin{cases} \text{the (subfactor) inclusion} \\ \mathrm{vN}_\pi(\mathbb{S}_{2,\infty}) \subset \mathrm{vN}_\pi(\mathbb{S}_\infty) \text{ is irreducible.} \end{cases}$

A simple application: Thoma multiplicativity

The limit cycles

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Remarks

- E_{-1} is a center-valued trace.
- If $\mathrm{vN}_\pi(\mathbb{S}_\infty)$ is a factor, then E_{-1} can be replaced by the tracial state tr:

$$\mathrm{tr}(\pi(\sigma)) = \prod_{k\geq 2} \big(\mathrm{tr}(A_0^{k-1})\big)^{m_k(\sigma)}$$

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Commuting squares & Discrete spectrum

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Corollary (Okounkov '97, Gohm & K '09)

Suppose $\mathrm{vN}_\pi(\mathbb{S}_\infty)$ is a factor. Then the limit 2-cycle $A_0 = E_0(v_1)$ has discrete spectrum which may only accumulate at the point 0.

Thoma measures

Definition

A discrete probability measure μ on $[-1,1]$ satisfying

$$\frac{\mu(t)}{|t|} \in \mathbb{N}_0 \qquad (t \neq 0)$$

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Suppose $\mathrm{vN}_\pi(\mathbb{S}_\infty)$ is a factor with tracial state tr. Then the spectral measure μ of the limit 2-cycle A_0 with respect to tr is a Thoma measure.

Conclusion: Thoma's theorem

The spectral measure μ is supported on the spectral values of A_0.

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for every $k > 1$. One recovers from this the traditional form of the Thoma theorem, by writing spectral values with multiplicities.

Noncommutative random measure factorizations

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- $F(1_B) = \mathcal{A}$
- $F(b)$ and $F(b')$ are $\mathcal{N}$-independent

A suitable continuity condition needs to be stipulated for a relevant class of sets $S \subset B$, if B is not finite. Here: S are all finite subsets of $\mathbb{N}$ and the continuity condition is $\bigvee_{s \in S} F(s) = \mathcal{A}$.

Factorizations from unitary representations of $\mathbb{S}_\infty$

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Then

$$F\colon 2^{\mathbb{N}} \to \mathcal{R}(\mathcal{A}, \varphi)$$

is a factorization of $(\mathcal{A}, \varphi)$ over $\mathcal{N}$.

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Thank you!