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PROXIMINALITY PROPERTIES IN $L_p(\mu, X)$ AND POLYHEDRAL DIRECT SUMS OF BANACH SPACES

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Abstract. For a closed subspace Y of a Banach space X, we define a separably determined property for Y in X. Let (P) be either proximinality or strong $1\frac{1}{2}$-ball property and if (P) is separably determined for Y in X, then we prove that $L_1(\mu, Y)$ has the same property (P) in $L_1(\mu, X)$. For an M-embedded space X, we give a class of elements in $L_1(\mu, X^{**})$ having best approximations from $L_1(\mu, X)$. We also prove that some of these proximinality properties are stable under polyhedral direct sums of Banach spaces.

1. Introduction

Let X be a Banach space and Y be a closed subspace of X. For a non-empty subset K of X and $x \in X$, the distance of x from K, denoted by $d(x, K)$, is given by $d(x, K) = \inf\{\|x - k\| : k \in K\}$. Recall that the metric projection from X onto Y is the set valued map defined by $P_Y(x) = \{y \in Y : d(x, Y) = \|x - y\|\}$ and Y is said to be proximinal in X if $P_Y(x) \neq \emptyset$ for all $x \in X$. A subspace Y is said to be ball proximinal in X if for every $x \in X$, there exists an element $y \in B_Y$ such that $d(x, B_Y) = \|x - y\|$, where B_Y is the closed unit ball of Y(See [1] for details).

In this paper, we restrict ourselves to real scalars and all subspaces we consider are assumed to be closed. For $x \in X$ and $r > 0$, let $B(x, r)$ denote the closed ball of radius r centered at x. Even for balls in subspaces, we use the same notation without explicitly mentioning the subspace. The closed unit ball and the unit sphere of X will be denoted by B_X and S_X respectively. Under the canonical embedding we will consider X as a subspace of X^{**}.

The following stronger version of proximinality was introduced in [5].

Definition 1.1. A proximinal subspace Y of X is said to be strongly proximinal if for every $x \in X$ and every $\epsilon > 0$, there exists a $\delta > 0$ such that $P_Y(x, \delta) \subseteq P_Y(x) + \epsilon B_X$, where $P_Y(x, \delta) = \{y \in Y : \|x - y\| < d(x, Y) + \delta\}$.

The following notion of differentiability defined in [4] characterizes a strongly proximinal hyperplane.

Definition 1.2. The norm on X is strongly subdifferentiable (SSD in short) at $x \in X$ if the one sided limit

$$d^+(x)(y) := \lim_{t \to 0^+} \frac{\|x + ty\| - \|x\|}{t}$$

exists uniformly for $y \in S_X$. In this case, we call x is an SSD point of X.

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In [5], it was proved that for an $f \in S_{X^*}$, $ker(f)$ is strongly proximinal if and only if f is an SSD-point of X^*.
The following notion of intersection property of balls was introduced in [16] which is also a stronger version of proximinality (in fact, stronger than strong proximinality).

Definition 1.3. A subspace Y of a Banach space X is said to have the (strong)$1\frac{1}{2}$-ball property if the conditions $x \in X$, $y \in Y$, $Y \cap B(x, r) \neq \emptyset$ and $\|x - y\| \leq r + s$ $(r, s > 0)$ implies that $Y \cap B(x, r + \epsilon) \cap B(y, s + \epsilon) \neq \emptyset$ for all $(\epsilon \geq 0)\epsilon > 0$.

Theorem 2.11 of [7] shows that a subspace Y of a Banach space X has strong $1\frac{1}{2}$-ball property if and only if Y has $1\frac{1}{2}$-ball property and is ball proximinal in X. Also, a subspace Y has strong $1\frac{1}{2}$-ball property in X if and only if for every $x \in X$, there exists an element $y \in P_Y(x)$ such that $\|x\| = \|x - y\| + \|y\|$ ([11, Proposition 3(ii)]).
A further strengthening of the $1\frac{1}{2}$-ball property is the following well-studied concept called M-ideal(see [6, Chapter 1 and Chapter 2] for more details).

Definition 1.4 ([6])**.** A subspace Y of a Banach space X is an M-ideal in X if there is a linear projection P on X^* with $ker(P) = Y^{\perp}$ and for all $x^* \in X^*$, $\|x^*\| = \|Px^*\| + \|x^* - Px^*\|$. A Banach space X is called an M-embedded space if X is an M-ideal in X^{**}.

For a complete positive σ-finite measure space (Ω, Σ, μ), we denote by $L_p(\mu, X)$ the Banach space of all Bochner p-integrable(essentially bounded for $p = \infty$) functions on Ω with values in X, endowed with the usual p-norm (see [2] for details).

In approximation theory, one of the important problems is the following: Suppose that a subspace Y of a Banach space X has some proximinality property(viz. proximinality, strong proximinality, $1\frac{1}{2}$-ball property, strong $1\frac{1}{2}$-ball property, M-ideal etc) in X. Does it follows that $L_1(\mu, Y)$ has the same proximinality property in $L_1(\mu, X)$? Since, for a measurable set E with $\mu(E) > 0$, the map $P : L_1(\mu, X) \to L_1(\mu, X)$ defined by $P(f) = f\chi_E$ is a nontrivial L-projection, by [6, Theorem 1.8], $L_1(\mu, X)$ cannot have any M-ideal provided that the dimension of $L_1(\mu, X)$ is greater than 2. Therefore, if the dimension of $L_1(\mu, X)$ is greater than 2, then $L_1(\mu, Y)$ can never be an M-ideal in $L_1(\mu, X)$. Then under the assumption of Y to be an M-ideal one can ask about the strongest proximinality property that $L_1(\mu, Y)$ possess. Proposition 2.10 gives a partial answer to this.

In [10], José Mendoza proved that $L_1(\mu, Y)$ is not proximinal in $L_1(\mu, X)$ even if Y is proximinal in X but $L_1(\mu, Y)$ is proximinal if Y is a separable proximinal subspace. For a non-atomic σ-finite countably generated measure space, an analogous result for strong $1\frac{1}{2}$-ball property is proved in [12]. In Section 2, we prove these two results for every non-separable subspace Y of X satisfying a general condition: "each separable subspace of Y is contained in a separable subspace of Y that has the appropriate proximinality property in X ". In the same section, we also give a class of Banach spaces and subspaces where this general condition hold. But if the proximinality property under consideration is strong proximinality or $1\frac{1}{2}$-ball property, an analogous result for the above problem is not known.

Now moving to the discrete version of the above problem, one can ask for the stability of these proximinality properties under ℓ_p-sums$(1 \leq p \leq \infty)$ and c_0-sums of Banach spaces, where ℓ_p-sums$(1 \leq p \leq \infty)$ and c_0-sums of Banach spaces are defined as follows:

Definition 1.5. For an arbitrary collection $\{X_i : i \in I\}$ of Banach spaces,

(a) ℓ_p-sum $(1 \leq p < \infty)$ of X_i $(i \in I)$ is defined by

$$\bigoplus_p X_i = \{x \in \prod X_i : \|x\| = (\sum \|x(i)\|^p)^{1/p}\}.$$

(b) ℓ_∞-sum of X_i $(i \in I)$ is defined by

$$\bigoplus_\infty X_i = \{x \in \prod X_i : \|x\| = \sup \|x(i)\| < \infty\}.$$

(c) c_0-sum of X_i $(i \in I)$ is defined by

$$\bigoplus_{c_0} X_i = \{x \in \prod X_i : \{i \in I : \|x(i)\| > \epsilon\} \text{ is finite for all } \epsilon > 0\}$$

with sup norm on it.

In the discrete version, we ask the following question: Suppose that for each $i \in I$, Y_i is a subspace of X_i having some proximinality property in X_i and let $1 \leq p \leq \infty$. Does this implies that $\bigoplus_p Y_i$ and $\bigoplus_{c_0} Y_i$ has the same property in $\bigoplus_p X_i$ and $\bigoplus_{c_0} X_i$ respectively? The answer is affirmative if the property under consideration is proximinality. In [12], it was proved that the $1\frac{1}{2}$-ball property and strong $1\frac{1}{2}$-ball property are stable under ℓ_∞-sums and c_0-sums. It was also proved in [12] that the $1\frac{1}{2}$-ball property is stable under ℓ_1-sums. But the stability of strong $1\frac{1}{2}$-ball property under ℓ_1-sums is still not known.

It is proved in [8] that the strong proximinality is stable under c_0-sums and finite ℓ_∞-sums of Banach spaces. In section 3, we prove that the strong proximinality is stable under finite ℓ_1-sums also. In fact, we prove that proximinality and strong proximinality(under an additional assumption) are stable under a recently introduced concept called polyhedral direct sums of Banach spaces([14, Definition 2.1]).

[2] and [6] are standard references for any unexplained terminology.

2. Separably Determined Properties

We begin this section by defining a separably determined property which plays a major role in this section.

Definition 2.1. Let X be a non-separable Banach space and let (P) be a property in X. (P) is called a separably determined property for Y in X if for every separable subspace Z of Y, there exists a separable subspace Z' such that $Z \subseteq Z' \subseteq Y$ and Z' has the property (P) in X.

For some of the proximinality property (P), if (P) is separably determined for Y in X, then Y has the property (P) in X. We prove this in our next theorem.

Theorem 2.2. *Let Y be a non-separable subspace of a non-separable Banach space X and let* (P) *be one of the following properties:*

(a) *Proximinality.*
(b) *Ball proximinality*
(c) *Strong proximinality.*
(d) $1\frac{1}{2}$*-Ball property.*
(e) *Strong* $1\frac{1}{2}$*-Ball property.*

If (P) *is separably determined for Y in X, then Y has the property* (P) *in X.*

Proof. (a) Let $x \in X$. Suppose (y_n) is a sequence in Y such that $d(x, Y) = \lim \|x - y_n\|$. Now let $Z = \overline{\text{span}}\{y_n\}_{n\geq 1}$. Then there exists a separable space Z' such that $Z \subseteq Z' \subseteq Y$ and Z' is proximinal in X. Since Z' is proximinal, there exists an element $z' \in Z'$ such that $d(x, Z') = \|x - z'\|$ and hence $\|x - z'\| = d(x, Z') \leq \lim \|x - y_n\| = d(x, Y) \leq d(x, Z') = \|x - z'\|$. This implies that $d(x, Y) = \|x - z'\|$ and therefore Y is proximinal in X.

(b) Proof of this is similar to that of (a).

(c) Suppose Y is not strongly proximinal in X. Then there exists an $\varepsilon > 0$ such that for all $n \in \mathbb{N}$, there exists an element $y_n \in P_Y(x, \frac{1}{n})$ such that $d(y_n, P_Y(x)) > \varepsilon$. Let $Z = \overline{\text{span}}\{y_n\}$. Then by assumption, there exists a separable space Z' such that $Z \subseteq Z' \subseteq Y$ and Z' is strongly proximinal in X. Therefore there exists a $\delta > 0$ such that $P_{Z'}(x, \delta) \subseteq P_{Z'}(x) + \varepsilon B_Y$. Since $y_n \in P_Y(x, \frac{1}{n})$, we get $d(x, Y) = d(x, Z') = \lim \|x - y_n\|$ which in turn implies $P_{Z'}(x) \subseteq P_Y(x)$. Hence $d(y_n, P_{Z'}(x)) \geq d(y_n, P_Y(x)) > \varepsilon$ for all n. But since $\|x - y_n\|$ converges to $d(x, Z')$, $\|x - y_n\| < d(x, Z') + \delta$ for sufficiently large n. Hence $d(y_n, P_{Z'}(x)) \leq \varepsilon$. This contradiction proves (b).

(d) Let $x \in X,\ y \in Y,\ B(x, r) \cap Y \neq \emptyset$ and $\|x - y\| \leq r + s$ $(r,\ s > 0)$. Let $y_0 \in B(x, r) \cap Y$ and $Z = \text{span}\{y, y_0\}$. Then by assumption, there exists a separable space Z' such that $Z \subseteq Z' \subseteq Y$ and Z' has $1\frac{1}{2}$-ball property in X. Hence $B(x, r + \varepsilon) \cap B(y, s + \varepsilon) \cap Y \neq \emptyset$ for all $\varepsilon > 0$.

(e) Proof of this is similar to that of (d).

□

Now we will give examples of Banach spaces X and subspaces Y such that proximinality is separably determined for Y in X. Since subspace of a reflexive space is proximinal, we will use reflexive spaces to produce such examples.

Recall that a subspace Y of a Banach space X is a said to be a factor reflexive subspace if the quotient space X/Y is reflexive.

Lemma 2.3. *Let $\{X_i : i \in \mathbb{N}\}$ be a countable family of reflexive spaces and let $X = \bigoplus_{c_0} X_i$. Then for every separable subspace Y of X, there exists a separable proximinal subspace Z of X such that $Y \subseteq Z \subseteq X$.*

Proof. Since Y is separable, there exists a countable set $\{y_n\} \subseteq Y$ such that $Y = \overline{\text{span}}\{y_n\}$. Let $Z_i = \overline{\text{span}}\{y_n(i) : n = 1, 2, \ldots\}$ and $Z = \bigoplus_{c_0} Z_i$. Clearly $Z \subseteq Y \subseteq X$. Since each Z_i is a separable proximinal subspace of X_i, Z is a separable proximinal subspace of X. □

Theorem 2.4. *Let $\{X_i : i \in I\}$ be a family of reflexive spaces, $X = \bigoplus_{c_0} X_i$ and Y a proximinal factor reflexive subspace of X. Then proximinality is separably determined for Y in X.*

Proof. Since Y is a proximinal factor reflexive subspace of X, by Garkavi's theorem([13, Chapter III, Lemma 1.1]), all $f \in Y^{\perp}$ is norm attaining. Hence there exists an $x \in S_X$ such that $f(x) = 1 = \|f\|$. Since $f \in X^* = \bigoplus_1 X_i^*$, $f(i)(x(i)) = \|f(i)\|$.

Now suppose $f(i) \neq 0$ for infinitely many i. Then for these infinitely many i, we have

$$1 = \frac{f(i)}{\|f(i)\|}(x(i)) = \left|\frac{f(i)}{\|f(i)\|}(x(i))\right| \leq \|x(i)\|.$$

which contradicts the fact that $x \in \bigoplus_{c_0} X_i$. Hence $f(i) = 0$ for all but finitely many i. Then by Baire category theorem, there exists a finite subset A of I such that $f(i) = 0$ for all $f \in Y^\perp$ and $i \notin A$. Then by a canonical identification, $Y^\perp \subseteq \bigoplus_{1, i\in A} X_i^*$. Hence

$$Y = \left(Y \cap \bigoplus_{\infty, i \in A} X_i\right) \bigoplus\nolimits_\infty \left(\bigoplus_{c_0, i\notin A} X_i\right).$$

Let Z be a separable subspace of Y and $\{z_n\}_{n\in\mathbb{N}} \subseteq Z$ be such that $Z = \overline{\text{span}}\{z_n\}_{n\in\mathbb{N}}$. Then for every $n \in \mathbb{N}$, there exists an element $v_n \in Y \cap \bigoplus_{\infty, i\in A} X_i$ and $w_n \in \bigoplus_{c_0, i\notin A} X_i$ such that $z_n = v_n + w_n$ and $\|z_n\| = \max\{\|v_n\|, \|w_n\|\}$. Now let $V = \overline{\text{span}}\{v_n\}_{n\in\mathbb{N}}$ and $W = \overline{\text{span}}\{w_n\}_{n\in\mathbb{N}}$. Clearly $V \subseteq Y \cap \bigoplus_{\infty, i\in A} X_i$. Since A is finite, V is a separable proximinal subspace of $\bigoplus_{\infty, i\in A} X_i$. Since W is a separable subspace of $\bigoplus_{c_0, i\notin A} X_i$, there exists a countable subset A_0 of $I \setminus A$ such that $W \subseteq \bigoplus_{\infty, i\in A_0} X_i$. Then, by Lemma 2.3, there exists a separable proximinal subspace W' of $\bigoplus_{\infty, i\in A_0} X_i$ such that $W \subseteq W' \subseteq \bigoplus_{\infty, i\in A_0} X_i$. Since $\bigoplus_{\infty, i\in A_0} X_i$ is an M-summand in $\bigoplus_{\infty, i\notin A} X_i$, W' is proximinal in $\bigoplus_{c_0, i\notin A} X_i$. Then $Z' = V \bigoplus_\infty W'$ is the required subspace of X. $\square$

Corollary 2.5. *Let Y be a finite co-dimensional proximinal subspace of $c_0(I)$, where I is an arbitrary indexing set. Then proximinality is separably determined for Y in $c_0(I)$.*

A proof similar to that of Lemma 2.3 gives

Lemma 2.6. *Let $\{X_i : i \in \mathbb{N}\}$ be a countable family of reflexive spaces and let $X = \bigoplus_1 X_i$. Then for every separable subspace Y of X, there exists a separable proximinal subspace Z of X such that $Y \subseteq Z \subseteq X$.*

Lemma 2.6 along with similar arguments used in the proof of Theorem 2.4 gives

Theorem 2.7. *Let $\{X_i : i \in I\}$ be any family of reflexive spaces, $X = \bigoplus_1 X_i$ and Y be a subspace of X such that there exists a finite subset A of I such that $f(i) = 0$ for all $f \in Y^\perp$ and $i \notin A$. Then proximinality is separably determined for Y in X.*

Our next theorem generalizes Corollary 3.5 of [10]. Even though it is noted in [10, Remark 3.6], we give a proof of it for the sake of completeness.

Theorem 2.8. *Let (Ω, Σ, μ) be a complete positive σ-finite measure space. Let X be a Banach space and Y be a subspace of X such that proximinality is separably determined for Y in X. Then $L_1(\mu, Y)$ is proximinal in $L_1(\mu, X)$.*

Proof. Let $f \in L_1(\mu, X)$. Let (f_n) be a sequence in $L_1(\mu, Y)$ such that $d(f, L_1(\mu, Y)) = \lim \|f - f_n\|_1$. Since $f_n's$ are μ-essentially separably valued, without loss of generality, we can assume that range(f_n) is separable for all n. Now let $Z_n = \overline{\text{range}(f_n)}$ and $Z = \overline{\text{span}}\{\cup Z_n\}$. Since Z is a separable subspace of Y, there exists a separable proximinal subspace Z' of X such that $Z \subseteq Z' \subseteq Y$. Then, by [10, Theorem 3.4], $L_1(\mu, Z')$ is proximinal in $L_1(\mu, X)$. Hence there exists an element $g \in L_1(\mu, Z')$ such that $\|f - g\|_1 = d(f, L_1(\mu, Z'))$. Then

$$d(f, L_1(\mu, Y)) \leq \|f - g\|_1 = d(f, L_1(\mu, Z')) \leq \lim_{n\to\infty} \|f - f_n\|_1 = d(f, L_1(\mu, Y)).$$

Therefore $L_1(\mu, Y)$ is proximinal in $L_1(\mu, X)$. □

Our next theorem generalizes Theorem 15 of [12].

Theorem 2.9. *Let (Ω, Σ, μ) be a non-atomic σ-finite countably generated measure space and let $1 \leq p \leq \infty$. Let X be a Banach space and Y be a subspace of X such that strong $1\frac{1}{2}$-ball property is separably determined for Y in X. Then $L_1(\mu, Y)$ has strong $1\frac{1}{2}$-ball property in $L_1(\mu, X)$.*

Proof. Let $f \in L_1(\mu, X)$. Let (f_n) be a sequence in $L_1(\mu, Y)$ such that $d(f, L_1(\mu, Y)) = \lim \|f - f_n\|_1$. Since $f_n's$ are μ-essentially separably valued, without loss of generality, we can assume that range(f_n) is separable for all n. Now let $Z_n = \overline{\text{range}(f_n)}$ and $Z = \overline{\text{span}}\{\cup Z_n\}$. Since Z is separable subspace of Y, there exists a separable subspace Z' of X such that $Z \subseteq Z' \subseteq Y$ and Z' has strong $1\frac{1}{2}$-ball property in X. Then, by [12, Theorem 15], $L_1(\mu, Z')$ has strong $1\frac{1}{2}$-ball property in $L_1(\mu, X)$. Hence there exists an element $g \in P_{L_1(\mu, Z')}(f)$ such that $\|f\|_1 = \|f - g\|_1 + \|g\|_1$. Since $L_1(\mu, Z') \subseteq L_1(\mu, Y)$,

$$d(f, L_1(\mu, Y)) \leq \|f - g\|_1 = d(f, L_1(\mu, Z')) \leq \lim_{n\to\infty} \|f - f_n\|_1 = d(f, L_1(\mu, Y)).$$

Hence $g \in P_{L_1(\mu, Y)}(f)$ and the result follows. □

Combining [7, Theorem 2.11] and [12, Theorem 15], we can easily observe that if Y is a separable ball proximinal subspace of X having $1\frac{1}{2}$-ball property in X, then $L_1(\mu, Y)$ has strong $1\frac{1}{2}$-ball property in $L_1(\mu, X)$. But even for a separable M-ideal Y in X, it is not known that whether $L_1(\mu, Y)$ has strong $1\frac{1}{2}$-ball property or not. Now since M-embedded spaces are 'weakly compactly generated', we can find a class of elements in $L_1(\mu, X^{**})$ having best approximations from $L_1(\mu, X)$. Our next result proves this.

Recall that a Banach space is called weakly compactly generated if it is the closed linear span of some weakly compact set. M-embedded spaces are weakly compactly generated (see [6, Page 142] for more details).

In [7], it is wrongly stated that an M-ideal is ball proximinal. Example 13 of [15] shows that an M-ideal may not be ball proximinal. Therefore we add ball proximinality as an additional assumption in our next result.

Proposition 2.10. *Let X be an M-embedded space and X be ball proximinal in X^{**}. Let $f \in L_1(\mu, X^{**})$ be such that range$(f) \subseteq Z^{\perp\perp}$, where Z is a separable subspace of X. Then there exists an $f_0 \in L_1(\mu, X)$ such that $d(f, L_1(\mu, X)) = \|f - f_0\|$ and $\|f\| = \|f - f_0\| + \|f_0\|$.*

Proof. Let $f \in L_1(\mu, X^{**})$ and Z be a separable subspace of X such that $\text{range}(f) \subseteq Z^{\perp\perp}$. Since X is an M-embedded space, by [6, Chapter III, Theorem 4.6] and [3, Chapter 5, Section 2, Theorem 3], there exists a separable subspace Y of X such that $Z \subseteq Y \subseteq X$ and a projection $P : X \to Y$ such that $\|P\| = 1$. Then Y is ball proximinal in $Y^{\perp\perp}$. For, let $\varphi \in Y^{\perp\perp}$. Since X is ball proximinal in X^{**}, there exists an element $x \in B_X$ such that $d(\varphi, B_X) = \|\varphi - x\|$. Then $P(x) \in B_Y$ and

$$d(\varphi, B_Y) \geq d(\varphi, B_X) \geq \|\varphi - x\| \geq \|P^{**}(\varphi - x)\| = \|\varphi - P(x)\| \geq d(\varphi, B_Y).$$

Hence Y is ball proximinal in $Y^{\perp\perp}$. Since, by [6, Chapter III, Theorem 1.6(a)], subspace of an M-embedded space is an M-embedded space, by [12, Theorem 15], $L_1(\mu, Y)$ has strong $1\frac{1}{2}$-ball property in $L_1(\mu, Y^{**})$. Hence there exists an element $f_0 \in L_1(\mu, Y)$ such that $d(f, L_1(\mu, Y)) = \|f - f_0\|$ and $\|f\| = \|f - f_0\| + \|f_0\|$. Now let $g \in L_1(\mu, X)$. Then

$$\|f(t) - g(t)\| \geq \|P^{**}(f(t) - g(t))\| = \|f(t) - P(g(t))\|.$$

Hence $\|f - g\| \geq \|f - P \circ g\| \geq \|f - f_0\|$. Therefore $f_0 \in P_{L_1(\mu,X)}(f)$ and the result follows. ☐

For a probability measure μ, our next theorem gives a necessary condition for $L_p(\mu, Y)$ to be strongly proximinal in $L_p(\mu, X)$.

Theorem 2.11. *Let Y be a subspace of a Banach space X. Let (Ω, Σ, μ) be a probability space and let $1 \leq p \leq \infty$. If $L_p(\mu, Y)$ is strongly proximinal in $L_p(\mu, X)$, then Y is strongly proximinal in X.*

Proof. Suppose $L_p(\mu, Y)$ is strongly proximinal in $L_p(\mu, X)$. Let $x \in X$ and $\varepsilon > 0$. Define $f \in L_p(\mu, X)$ as $f = x\chi_\Omega$. Then there exists a $\delta > 0$ such that

$$P_{L_p(\mu,Y)}(f, \delta) \subset P_{L_p(\mu,Y)}(f) + \varepsilon B_{L_p(\mu,X)}.$$

Now let $y \in P_Y(x, \delta)$. Define $g \in L_p(\mu, Y)$ as $g = y\chi_\Omega$. Then $\|x - y\| = \|f - g\|_p$.

Case 1: $1 \leq p < \infty$.

For $h \in L_p(\mu, Y)$,

$$\|f - h\|_p^p = \int_\Omega \|f(\omega) - h(\omega)\|^p \, d\mu \geq \int_\Omega d(f(\omega), Y)^p \, d\mu = d(x, Y)^p.$$

Hence $d(x, Y) \leq d(f, L_p(\mu, Y))$. Therefore $g \in P_{L_p(\mu,Y)}(f, \delta)$. Then, by assumption, there exists an element $g' \in P_{L_p(\mu,Y)}(f)$ such that $\|g - g'\|_p \leq \varepsilon$. Now put $y_0 = \int_\Omega g' \, d\mu$. Then, for all $h \in L_p(\mu, Y)$,

$$\|x - y_0\| = \left\| \int_\Omega f \, d\mu - \int_\Omega g' \, d\mu \right\| \leq \|f - g'\|_p = d(f, L_p(\mu, Y)) \leq \|f - h\|_p.$$

Now for $u \in Y$, define $h' \in L_p(\mu, Y)$ as $h' = u\chi_\Omega$. Then $\|x - y_0\| \leq \|f - h'\|_p = \|x - u\|$. Hence $y_0 \in P_Y(x)$. Since $\|y - y_0\| \leq \|g - g'\|_p \leq \varepsilon$, $y \in P_Y(x) + \varepsilon B_X$. This completes the proof for $1 \leq p < \infty$.

Case 2: $p = \infty$

For $h \in L_\infty(\mu, Y)$, since $\|f(\omega) - h(\omega)\| \leq \|f - h\|_\infty$ for almost all $\omega \in \Omega$, $d(x, Y) \leq \|f - h\|_\infty$. Hence $d(x, Y) \leq d(f, L_\infty(\mu, Y))$. Therefore $g \in P_{L_\infty(\mu,Y)}(f, \delta)$. Then, by assumption, there exists an element $g' \in P_{L_\infty(\mu,Y)}(f)$ such that $\|g-g'\|_\infty \leq \varepsilon$. Hence there exists a measure zero set E such that $\|f(\omega)-g'(\omega)\| \leq d(f, L_\infty(\mu,Y))$ and $\|g(\omega)-g'(\omega)\| \leq \varepsilon$ for all $\omega \notin E$. Fix an $\omega_0 \notin E$ and define $y_0 \in Y$ as $y_0 = g'(\omega_0)$. Now for $u \in Y$, define $h' \in L_p(\mu, Y)$ as $h' = u\chi_\Omega$. Then $\|x-y_0\| \leq \|f-h'\|_\infty = \|x-u\|$. Hence $y_0 \in P_Y(x)$. Since $\|y-y_0\| \leq \|g(\omega_0)-g'(\omega_0)\| \leq \varepsilon$, $y \in P_Y(x)+\varepsilon B_X$. This completes the proof for $p=\infty$. ☐

But the converse of the Theorem 2.11 is still not known.

Question 2.12. Let (Ω, Σ, μ) be a probability space and Y be a strongly proximinal subspace of a Banach space X. Let $1 \leq p \leq \infty$. Is $L_p(\mu, Y)$ strongly proximinal in $L_p(\mu, X)$?

3. Stability of proximinality properties under direct sums

We begin this section with two well-known lemmas which describe the distance function in ℓ_p-sums and ℓ_∞-sums of Banach spaces.

Lemma 3.1. *Let $\{X_i : i \in I\}$ be a family of Banach spaces and Y_i be a proximinal subspace of X_i. Let $1 \leq p < \infty$. Let $X = \bigoplus_p X_i$ and $Y = \bigoplus_p Y_i$. Then for $x = (x(i)) \in X$, $d(x,Y) = \left(\sum_{i\in I} d(x(i), Y_i)^p\right)^{1/p}$.*

Proof. Since each Y_i is proximinal in X_i, there exists an element $y_i' \in Y_i$ such that $d(x(i), Y_i) = \|x(i) - y_i'\|$. Define $y' \in Y$ as $y'(i) = y_i'$. Since $\|x(i) - y'(i)\| \leq \|x(i)\|$ for all i, $y \in \bigoplus_p Y_i$. Then $d(x,Y)^p \leq \sum_{i\in I} \|x(i)-y'(i)\|^p = \sum_{i\in I} d(x(i), Y_i)^p$. Now for any $y \in Y$, $\sum_{i\in I} d(x(i), Y_i)^p \leq \sum_{i\in I} \|x(i)-y(i)\|^p \leq \|x-y\|^p$. Hence the lemma follows. ☐

Similarly we can prove that

Lemma 3.2 ([9])**.** *Let $\{X_i : i \in I\}$ be a family of Banach spaces and Y_i be a proximinal subspace of X_i. Let $X = \bigoplus_\infty X_i$ $(X = \bigoplus_{c_0} X_i)$ and $Y = \bigoplus_\infty Y_i$ $(Y = \bigoplus_{c_0} Y_i)$. Then for $x \in X$, $d(x,Y) = \sup_{i\in I} d\left(x(i), Y_i\right)$.*

As an immediate consequence of Lemma 3.1 and Lemma 3.2, we have the following:

Theorem 3.3. *Let $\{X_i : i \in I\}$ be a family of Banach spaces and Y_i be a subspace of X_i. Let $1 \leq p \leq \infty$. Then the following are equivalent:*

(i) *Y_i is proximinal in X_i for all $i \in I$.*
(ii) *$\bigoplus_p Y_i$ is proximinal in $\bigoplus_p X_i$.*
(iii) *$\bigoplus_{c_0} Y_i$ is proximinal in $\bigoplus_{c_0} X_i$.*

In [14], Libor Veselý defined a new direct sum called polyhedral direct sum of Banach spaces. Now we prove the stability of some proximinality properties under polyhedral direct sums.

Definition 3.4 ([14])**.** A function $\pi : \mathbb{R}_+^n \to \mathbb{R}_+$ is a norm on $\mathbb{R}_+^n$ if it is subadditive, positively homogeneous and $\pi(t) \neq 0 \Leftrightarrow t = 0$. A norm π on $\mathbb{R}_+^n$ is called polyhedral if it is of the form $\pi(t) = \max_{1\leq i\leq m} g_j(t)$, where $g_1, ..., g_m \in (\mathbb{R}^n)^*$. In this case, we say that the

family $\{g_1, ..., g_m\}$ generates π. Note that if $\{g_1, \ldots, g_m\}$ is a minimal family generating π, then $g_j(i) \geq 0$ for all $i = 1, \ldots, n$ and $j = 1, \ldots, m$.
We say that a Banach space X is the polyhedral direct sum of Banach spaces $X_1, ..., X_n$ if $X = X_1 \bigoplus ... \bigoplus X_n$ and the norm on X is of the form $\|x\|_X = \pi(\|x(1)\|, ..., \|x(n)\|)$, where π is a polyhedral nondecreasing norm on $\mathbb{R}^n_+$. In this case, we write $X = (X_1 \bigoplus ... \bigoplus X_n)_\pi$.

Our next theorem proves the stability of proximinality under polyhedral direct sums. For $i = 1, 2, \ldots, n$, $e_i \in \mathbb{R}^n$ is defined by $e_i(j) = 0$ if $i \neq j$ and $e_i(j) = 1$ if $i = j$.

Theorem 3.5. *Let X be a polyhedral direct sum of Banach spaces X_i $(1 \leq i \leq n)$ and Y_i be a subspace of X_i $(1 \leq i \leq n)$. Let π be the corresponding polyhedral norm and let $\pi(e_i) \neq 0$ for all i. Then the polyhedral direct sum Y of Y_i $(1 \leq i \leq n)$ is proximinal in X if and only if each Y_i is proximinal in X_i $(1 \leq i \leq n)$.*

Proof. Suppose that each Y_i is proximinal in X_i $(1 \leq i \leq n)$ and let $x \in X$.
Then there exists an element $y_i \in Y_i$ such that $\|x(i) - y_i\| = d(x(i), Y_i)$ $(1 \leq i \leq n)$. Now define $y \in Y$ as $y(i) = y_i$ $(1 \leq i \leq n)$. Then for $z \in Y$, we have $\|x(i)-y(i)\| \leq \|x(i)-z(i)\|$. Since π is monotone, $\|x - y\|_X \leq \|x - z\|_X$ for all $z \in Y$. Hence Y is proximinal in X.
Conversely suppose Y is proximinal in X and let $x_i \in X_i$.
Define $x \in X$ by

$$x(j) = \begin{cases} x_i & \text{if } j = i, \\ 0 & \text{otherwise.} \end{cases}$$

Then there exists an element $y \in Y$ such that $\|x-y\|_X = d(x, Y)$. Now, let $z_i \in Y_i$. Define $z \in X$ by

$$z(j) = \begin{cases} z_i & \text{if } j = i, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} \|x(i) - y(i)\|\pi(e_i) &\leq \|x - y\|_X \\ &\leq \|x - z\|_X \\ &= \|x(i) - z_i\|\pi(e_i). \end{aligned}$$

Hence Y_i is proximinal in X_i. □

Next lemma characterizes the distance function in polyhedral direct sums of Banach spaces.

Lemma 3.6. *Let X be the polyhedral direct sum of Banach spaces X_i $(1 \leq i \leq n)$ with π as the corresponding polyhedral norm. Let Y_i be a proximinal subspace of X_i $(1 \leq i \leq n)$ and Y be the polyhedral direct sum of Y_i $(1 \leq i \leq n)$. Then for $x \in X$,*

$$d(x, Y) = \pi(d(x(1), Y_1), \ldots, d(x(n), Y_n))$$

Proof. Let $y'_i \in Y_i$ be such that $\|x(i)-y'_i\| = d(x(i),Y_i)$ for all i. Then

$$\begin{aligned} d(x,Y) &\leq \pi(\|x(1)-y'_1\|,\ldots,\|x(n)-y'_n\|) \\ &= \pi(d(x(1),Y_1),\ldots,d(x(n),Y_n)) \\ &\leq \pi(\|x(1)-y_1\|,\ldots,\|x(n)-y_n\|) \text{ for all } y_i \in Y_i \\ &= \|x-y\|_\pi \ \text{ for all } y \in Y. \end{aligned}$$

i.e. $d(x,Y) \leq \pi\left(d(x(1),Y_1),\ldots,d(x(n),Y_n)\right) \leq d(x,Y)$, which proves the lemma. ☐

Lemma 3.7. *Let X be a polyhedral direct sum of Banach spaces X_i $(1 \leq i \leq n)$ and Y_i be a subspace of X_i $(1 \leq i \leq n)$. Let Y be the polyhedral direct sum of Y_i $(1 \leq i \leq n)$. Then for $x \in X$, $P_{Y_1}(x(1)) \times \ldots \times P_{Y_n}(x(n)) \subseteq P_Y(x)$ and equality holds if $g_j(i) > 0$ for all $i = 1,\ldots,n$ and $j = 1,\ldots,m$, where $\{g_1,\ldots,g_m\}$ is a minimal family generating π.*

Proof. Let $y_i \in P_{Y_i}(x(i))$ for all i and $z \in Y$. Define $y \in Y$ as $y(i) = y_i$ $(1 \leq i \leq n)$. Then for any $z \in Y$, $\|x-y\|_\pi \leq \pi(\|x(1)-z(1)\|,\ldots,\|x(n)-z(n)\|) = \|x-z\|_\pi$. Hence $y \in P_Y(x)$.

Now suppose that $g_j(i) > 0$ for all $i = 1,\ldots,n$ and $j = 1,\ldots,m$. Let $y \in P_Y(x)$. Suppose that there exists an element $j \in \{1,\ldots,n\}$ such that $y(j) \notin P_{Y_j}(x(j))$. Without loss of generality, we can assume that $j = 1$. Now choose a $\delta > 0$ such that $\|y(1)-x(1)\| > d(x(1),Y_1)+\delta$ and let $r_1 = \min_j g_j(e_1)$. Then

$$\begin{aligned} \|x-y\|_\pi &= \pi\left(\|x(1)-y(1)\|,\ldots,\|x(n)-y(n)\|\right) \\ &\geq \pi\left(d(x(1),Y_1)+\delta, d(x(2),Y_2),\ldots,d(x(n),Y_n)\right) \\ &= \max_j g_j\left(d(x(1),Y_1)+\delta, d(x(2),Y_2),\ldots,d(x(n),Y_n)\right) \\ &\geq \max_j g_j\left(d(x(1),Y_1), d(x(2),Y_2),\ldots,d(x(n),Y_n)\right) + \delta r_1 \\ &= d(x,Y) + \delta r_1 \end{aligned}$$

which is a contradiction. ☐

Our next theorem shows that with an additional assumption on the polyhedral norm, strong proximinality is stable under polyhedral direct sums.

Theorem 3.8. *Let X be a polyhedral direct sum of Banach spaces X_i $(1 \leq i \leq n)$ and Y_i be a subspace of X_i $(1 \leq i \leq n)$. Let π be the polyhedral norm such that $g_j(i) > 0$ for all $i = 1,\ldots,n$ and $j = 1,\ldots,m$, where $\{g_1,\ldots,g_m\}$ is a minimal family generating π. Then polyhedral direct sum Y of Y_i $(1 \leq i \leq n)$ is strongly proximinal in X if and only if each Y_i is strongly proximinal in X_i $(1 \leq i \leq n)$.*

Proof. Suppose Y is strongly proximinal in X.
Let $x_i \in X_i$ and $\epsilon > 0$. Define $x \in X$ by

$$x(j) = \begin{cases} x_i & \text{if } j = i, \\ 0 & \text{otherwise.} \end{cases}$$

Then there exists a $\delta > 0$ such that $P_Y(x,\delta) \subseteq P_Y(x) + r_0\epsilon B_Y$, where $r_0 = \min\limits_{1\le i\le n} \pi(e_i)$.
Let $r = \max\limits_{1\le i\le n} \pi(e_i)$ and $y_i \in P_Y(x(i), \frac{\delta}{r})$.
Define $y \in Y$ by

$$y(j) = \begin{cases} y_i & \text{if } j = i, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\|x - y\|_\pi = \|x(i) - y(i)\|\pi(e_i) < d(x(i), Y_i)\pi(e_i) + \frac{\delta}{r}\pi(e_i) \le d(x, Y) + \delta$. i.e. $y \in P_Y(x, \delta)$. Then there exists an element $\widetilde{z} \in P_Y(x)$ such that $\|\widetilde{z} - y\| \le r_0\epsilon$. Hence $\widetilde{z}(i) \in P_{Y_i}(x(i))$ and $\|\widetilde{z}(i) - y(i)\|\pi(e_i) \le \|y - \widetilde{z}\| \le r_0\epsilon < \pi(e_i)\epsilon$. Therefore $P_{Y_i}(x_i, \frac{\delta}{r}) \subseteq P_{Y_i}(x_i) + \epsilon B_{Y_i}$ and hence Y_i is strongly proximinal in X_i.
Conversely suppose that each Y_i is strongly proximinal in X_i $(1 \le i \le n)$. Let $x \in X$ and $\epsilon > 0$. Then there exists a $\delta > 0$ such that $P_{Y_i}(x(i), \delta) \subseteq P_{Y_i}(x(i)) + \frac{\epsilon}{\pi(1)} B_{Y_i}$. Let $r_i = \min\limits_j g_j(e_i)$ and $r' = \min\limits_i r_i$. Let $y \in P_Y(x, \delta r')$. Then $y(i) \in P_{Y_i}(x(i), \delta)$ for all i. If not, then there exists an element $j \in \{1, \ldots, n\}$ such that $y(j) \notin P_{Y_j}(x(j), \delta)$. Without loss of generality, we can assume that $j = 1$. Then

$$\begin{aligned} \|x - y\|_\pi &= \pi(\|x(1) - y(1)\|, \ldots, \|x(n) - y(n)\|) \\ &\ge \pi(d(x(1), Y_1) + \delta, d(x(2), Y_2), \ldots, d(x(n), Y_n)\|) \\ &= \max_j g_j(d(x(1), Y_1) + \delta, d(x(2), Y_2), \ldots, d(x(n), Y_n)) \\ &\ge \max_j g_j(d(x(1), Y_1), d(x(2), Y_2), \ldots, d(x(n), Y_n)) + \delta r_1 \\ &\ge d(x, Y) + \delta r'. \end{aligned}$$

The above contradiction proves that $y(1) \in P_{Y_1}(x(1), \delta)$ and hence $y(i) \in P_{Y_i}(x(i), \delta)$ for all i. Then for every i, there exists an element $y'_i \in P_{Y_i}(x(i))$ such that $\|y(i) - y'_i\| \le \frac{\epsilon}{\pi(1)}$. Define a $y' \in Y$ as $y'(i) = y'_i$. Then $y' \in P_Y(x)$ and $\|y - y'\| = \pi(\|y(1) - y'(1)\|, \ldots, \|y(n) - y'(n)\|) \le \frac{\epsilon}{\pi(1)}\pi(1) = \epsilon$. Hence $y \in P_Y(x) + \epsilon B_X$ and the converse follows. ☐

In Theorem 3.8, by taking $X_i = \mathbb{R}$ $(1 \le i \le n)$ and $\pi(t) = g(t)$, where $g \in \mathbb{R}^n$ is given by $(1, 1, \ldots, 1)$, we have the following corollary.

Corollary 3.9. *Strong proximinality is stable under finite ℓ_1-sums.*

Since for an $f \in S_{X^*}$, $ker(f)$ is strongly proximinal in X if and only if f is an SSD-point of X^*, the problem of stability of strong subdifferentiability under the infinite sums of Banach spaces is of great importance. In [4], the stability of SSD-points under ℓ_p-sums $(1 < p \le \infty)$ and c_0-sums is discussed. Our next theorem characterizes SSD-points of ℓ_1-sums of dual spaces.

Theorem 3.10. *Let $\{X_i : i \in I\}$ be a family of Banach spaces and $X = \bigoplus_{c_0} X_i$. Then $f \in S_{X^*}$ is an SSD-point of X^* if and only if f has only finitely many non-zero components and for all i with $f(i) \neq 0$, $\frac{f(i)}{\|f(i)\|}$ is an SSD-point of X_i^*.*

Proof. It is well-known fact that $\left(\bigoplus_{c_0} X_i\right)^* = \bigoplus_1 X_i^*$. Now let $f \in S_{X^*}$ be an SSD-point of X^*. Since an SSD-point of X^* is norm attaining, there exists an element $x \in S_X$ such

that $f(x) = 1 = \|f\|$. Hence $f(i)(x(i)) = \|f(i)\|$.
Suppose $f(i) \neq 0$ for infinitely many i. Then for these infinitely many i, we have

$$1 = \frac{f(i)}{\|f(i)\|}(x(i)) = \left| \frac{f(i)}{\|f(i)\|}(x(i)) \right| \leq \|x(i)\|$$

which contradicts the fact that $x \in \bigoplus_{c_0} X_i$.
Now let A be a finite subset of I such that $f(i) \neq 0$ for $i \in A$ and $f(i) = 0$ for $i \notin A$. Now for $g \in B_{X^*}$ and $t > 0$,

$$\frac{\|f + tg\| - 1}{t} = \sum_{i \in A} \frac{\|f(i) + tg(i)\| - \|f(i)\|}{t} + \sum_{i \notin A} \|g(i)\|.$$

Now letting $t \to 0^+$ we get,

$$d^+(f)(g) = \sum_{i \in A} d^+(\tfrac{f(i)}{\|f(i)\|})(g(i)) + \sum_{i \notin A} g(i).$$

Hence for $g \in B_{X^*}$ and $t > 0$, we have

$$\frac{\|f + tg\| - 1}{t} - d^+(f)(g) = \sum_{i \in A} \left(\frac{\left\| \frac{f(i)}{\|f(i)\|} + \frac{t}{\|f(i)\|} g(i) \right\| - 1}{(\frac{t}{f(i)})} - d^+(\tfrac{f(i)}{\|f(i)\|})(g(i)) \right). \tag{3.1}$$

Now the necessity follows from the fact that

$$\frac{\left\| \frac{f(i)}{\|f(i)\|} + \frac{t}{\|f(i)\|} g(i) \right\| - 1}{(\frac{t}{f(i)})} - d^+(\tfrac{f(i)}{\|f(i)\|})(g(i)) \leq \tfrac{\|f+tg\|-1}{t} - d^+(f)(g)$$

for all $i \in A$.
Sufficiency part follows from the fact if $f \in S_{X^*}$ is such that $f(i) = 0$ for $i \notin A$, where A is a finite subset of I, then for every $g \in B_{X^*}$, (3.1) holds. ☐

Corollary 3.11. *Let I be an indexing set. Then SSD-points of $\ell_1(I)$ are precisely the finitely supported points in $S_{\ell_1(I)}$.*

Proceeding as in the proof of Theorem 3.10, we get

Theorem 3.12. *Let X_i $(1 \leq i \leq n)$ be Banach spaces and $X = \bigoplus_1 X_i$. Then $x \in S_X$ is an SSD-point of X if and only if for all i with $x(i) \neq 0$, $\frac{x(i)}{\|x(i)\|}$ is an SSD-point of X_i.*

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