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KO-groups of stunted complex and quaternionic projective spaces

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KO-GROUPS OF STUNTED COMPLEX AND QUATERNIONIC PROJECTIVE SPACES

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ABSTRACT. In this note we compute $\widetilde{KO}^i$-groups of the stunted projective space $\mathbb{F}\mathbb{P}^m/\mathbb{F}\mathbb{P}^n$, where $\mathbb{F}=\mathbb{C}$ or $\mathbb{H}$. We also prove some non-sectioning results of certain maps of stunted complex projective spaces into certain quotients.

1. INTRODUCTION

In this note we compute the $\widetilde{KO}^i$-groups of the stunted complex and quaternionic projective spaces. The $\widetilde{KO}^i$-groups of the stunted real projective spaces has been computed by Fujii and Yasui in [4].

We believe that the computations presented here are well-known to the experts, but not having found any reference in the literature we present the computations here.

The $\widetilde{KO}^i$-groups of the stunted complex projective spaces are given by the following theorem.

Theorem 1.1. *(1) Let $m \geq 1$ and $n \geq 0$. The groups $\widetilde{KO}^{4m+i}(\mathbb{C}\mathbb{P}^{2m+n}/\mathbb{C}\mathbb{P}^{2m-1})$ are as follows:*

$n \backslash i$	0	-1	-2	-3	-4	-5	-6	-7
$4r$	$\mathbb{Z}^{2r+1}$	$\mathbb{Z}_2$	$\mathbb{Z}^{2r} \oplus \mathbb{Z}_2$	0	$\mathbb{Z}^{2r+1}$	0	$\mathbb{Z}^{2r}$	0
$4r+1$	$\mathbb{Z}^{2r+1} \oplus \mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}^{2r+1} \oplus \mathbb{Z}_2$	0	$\mathbb{Z}^{2r+1}$	0	$\mathbb{Z}^{2r+1}$	$\mathbb{Z}_2$
$4r+2$	$\mathbb{Z}^{2r+2}$	$\mathbb{Z}_2$	$\mathbb{Z}^{2r+1} \oplus \mathbb{Z}_2$	0	$\mathbb{Z}^{2r+2}$	0	$\mathbb{Z}^{2r+1}$	0
$4r+3$	$\mathbb{Z}^{2r+2}$	$\mathbb{Z}_2$	$\mathbb{Z}^{2r+2} \oplus \mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}^{2r+2} \oplus \mathbb{Z}_2$	0	$\mathbb{Z}^{2r+2}$	0

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(2) If $m \geq 0$ and $n = 2r+1$ is odd, then we have

$$\widetilde{KO}^q(\mathbb{CP}^{2m+1+n}/\mathbb{CP}^{2m}) = \begin{cases} 0 & \text{if } q \text{ is odd} \\ \mathbb{Z}^{r+1} & \text{if } q \text{ is even} \end{cases}$$

(3) Let $m \geq 0$ and $n = 2r$ be even. The groups $\widetilde{KO}^q(\mathbb{CP}^{2m+1+n}/\mathbb{CP}^{2m})$ are as follows:

$m+r$ \ q	0	-1	-2	-3	-4	-5	-6	-7
even	$\mathbb{Z}^r \oplus \mathbb{Z}_2$	0	$\mathbb{Z}^{r+1}$	0	$\mathbb{Z}^r$	0	$\mathbb{Z}^{r+1}$	$\mathbb{Z}_2$
odd	$\mathbb{Z}^r$	0	$\mathbb{Z}^{r+1}$	$\mathbb{Z}_2$	$\mathbb{Z}^r \oplus \mathbb{Z}_2$	0	$\mathbb{Z}^{r+1}$	0

To describe the KO-theory of the stunted quaternionic projective spaces we introduce the following notations. Given integers $0 \leq n < m$ define

$$e := e(m,n) = \#\{p : 4n < p \leq 4m,\ p \equiv 0 \pmod 8\},$$
$$f := f(m,n) = \#\{p : 4n < p \leq 4m,\ p \equiv 0 \pmod 4\},$$
$$g := g(m,n) = \#\{p : 4n < p \leq 4m,\ p \equiv 4 \pmod 8\}$$

and

$$h := h(m,n) = \#\{p : 2n < p \leq 2m,\ p \equiv 0 \pmod 2\}.$$

We shall also make use of the notations

$$X_{m,n} = \mathbb{CP}^m/\mathbb{CP}^n;\ Y_{m,n} = \mathbb{HP}^m/\mathbb{HP}^n,$$

in the sequel for the stunted complex and quaternionic projective spaces respectively. The $\widetilde{KO}^i$-groups of the stunted quaternionic projective spaces are given by the following theorem.

Theorem 1.2. *Let $Y_{m,n}$ be as above with $0 \leq n < m$. Then*

(1) $\widetilde{KO}^0(Y_{m,n}) = \mathbb{Z}^f$.
(2) $\widetilde{KO}^{-1}(Y_{m,n}) = \mathbb{Z}_2^e$.
(3) $\widetilde{KO}^{-2}(Y_{m,n}) = \mathbb{Z}_2^e$.
(4) $\widetilde{KO}^{-3}(Y_{m,n}) = 0$.
(5) $\widetilde{KO}^{-4}(Y_{m,n}) = \mathbb{Z}^f$.
(6) $\widetilde{KO}^{-5}(Y_{m,n}) = \mathbb{Z}_2^g$.
(7) $\widetilde{KO}^{-6}(Y_{m,n}) = \mathbb{Z}_2^g$.
(8) $\widetilde{KO}^{-7}(Y_{m,n}) = 0$.

The proof of the above theorem involves analyzing the Atiyah-Hirzebruch spectral sequence in KO-theory for the stunted quaternionic projective space.

The complex K-theory of the stunted complex and quaternionic projective spaces can also be computed easily using the Atiyah-Hirzebruch spectral sequence. We state the result here without proof for completeness.

Theorem 1.3. *Let $0 \leq n < m$. Then*

(1)
$$\widetilde{K}^i(X_{m,n}) = \begin{cases} \mathbb{Z}^h & \text{if } i = 0 \\ 0 & \text{if } i = 1 \end{cases}$$

(2)
$$\widetilde{K}^i(Y_{m,n}) = \begin{cases} \mathbb{Z}^f & \text{if } i = 0 \\ 0 & \text{if } i = 1 \end{cases}$$

□

In Section 2 we prove Theorem 1.1 and prove some non-sectioning results of maps of certain stunted complex projective spaces to certain quotients. In Section 3 we prove Theorem 1.2.

2. Proof of Theorem 1.1

The proof of Theorem 1.1 will make use of the computations of the KO^i-groups of the complex projective spaces. These have been computed in various places (see, for example, [3, Theorem 2] and [2, Proposition 2.1]). We mention that Proposition 2.1 (ii) in [2] has a typographical error. The correct statement is given in Proposition 2.1 below.

Let ξ be a real vector bundle and let $T(\xi)$ denote its Thom space. It is well known (see, for example, [1, page 304]) that there is a homeomorphism

$$X_{m,n} = \mathbb{CP}^m/\mathbb{CP}^n \longrightarrow T((n+1)\xi)$$

where ξ the canonical bundle over $\mathbb{CP}^{m-n-1}$. Thus in particular, there is a homeomorphism

$$X_{2m+n,2m-1} \longrightarrow T(2m\xi)$$

where ξ is the canonical bundle over $\mathbb{CP}^n$.

For a rank $8k$ vector bundle ζ over a space X such that the structure group admits a reduction to $\mathrm{Spin}(8k)$, there is a Thom isomorphism

$$KO^{-i}(X) \longrightarrow \widetilde{KO}^{-i}(T(\zeta)).$$

Hence if ξ is the canonical bundle over $\mathbb{CP}^n$, then as $2m\xi \oplus \varepsilon^{4m}$ is spin, we have that

$$KO^{-i}(\mathbb{CP}^n) \cong \widetilde{KO}^{-i-4m}(X_{2m+n,2m-1}).$$

We now make use of the following proposition to obtain (1) of Theorem 1.1.

Proposition 2.1. ([2, Proposition 2.1])

(1)
$$\widetilde{KO}^q(\mathbb{CP}^{2n}) = \begin{cases} 0 & \textit{for } q \textit{ odd} \\ \mathbb{Z}^n & \textit{for } q \textit{ even.} \end{cases}$$

(2) $\widetilde{KO}^q(\mathbb{CP}^{2n-1}) \cong \widetilde{KO}^q(\mathbb{CP}^{2n-2}) \oplus KO^{q-4n+2}(\text{point})$. □

To obtain (2) and (3) of Theorem 1.1 we first note the following.

Proposition 2.2. *For each q, there is a short exact sequence*
$$0 \longrightarrow \widetilde{KO}^q(X_{2m+1+n,2m}) \longrightarrow \widetilde{KO}^q(\mathbb{CP}^{2m+1+n}) \longrightarrow \widetilde{KO}^q(\mathbb{CP}^{2m}) \longrightarrow 0.$$
This splits if q is even. If q is odd, then
$$\widetilde{KO}^q(X_{2m+1+n,2m}) \cong \widetilde{KO}^q(\mathbb{CP}^{2m+1+n}).$$

Proof. The cofiber sequence
$$\mathbb{CP}^{2m} \longrightarrow \mathbb{CP}^{2m+1+n} \longrightarrow X_{2m+1+n,2m}$$
gives rise to an exact sequence
$$\xrightarrow{\alpha} \widetilde{KO}^{q-1}(\mathbb{C}P^{2m}) \to \widetilde{KO}^q(X_{2m+1+n,2m}) \to \widetilde{KO}^q(\mathbb{CP}^{2m+1+n}) \xrightarrow{\alpha} \widetilde{KO}^q(\mathbb{CP}^{2m}) \to .$$
It follows from Theorem 2.1 of [8] that the homomorphism α is an epimorphism. This shows the existence of the short exact sequence in the statement of the proposition. If q is even, then by Proposition 2.1 the last group is free abelian and hence the exact sequence splits. If q is odd, then the last group is zero and we have the required isomorphism. This completes the proof of the proposition. □

The computations in cases (2) and (3) of Theorem 1.2 now follow from the above proposition and the Proposition 2.1. This completes the proof of the first theorem.

Remark 2.3. As was noted in the above proposition, the homomorphism
$$\alpha : \widetilde{KO}^q(\mathbb{CP}^{2m+1+n}) \xrightarrow{\alpha} \widetilde{KO}^q(\mathbb{CP}^{2m})$$
is an epimorphism. We mention that the homomorphism
$$\beta : \widetilde{KO}^q(\mathbb{CP}^{2m+n}) \longrightarrow \widetilde{KO}^q(\mathbb{CP}^{2m-1})$$
need not be an epimorphism. Indeed, if $2m+n=2$, $2m-1=1$ and $q=-7$, then the first group is zero and the second group is $\mathbb{Z}_2$.

We can use the computations in Theorem 1.1 to prove non-sectioning results. We illustrate some cases here. Consider the map
$$f := f_{k',k} : \mathbb{CP}^{2m-1+k'}/\mathbb{CP}^{2m-1} \longrightarrow \mathbb{CP}^{2m-1+k'}/\mathbb{CP}^{2m-1+k}$$

where $k < k'$ and the map

$$g := g_{k',k} : \mathbb{CP}^{2m+1+k'}/\mathbb{CP}^{2m} \longrightarrow \mathbb{CP}^{2m+1+k'}/\mathbb{CP}^{2m+k}$$

with $k \leq k'$.

Corollary 2.4. *Let f and g be the maps defined above.*

(1) *Assume that m is odd and $k = 2t$ with t odd. Then f does not have a section.*
(2) *Assume that $k = 2t - 1$ is odd and $(m + t)$ is even. Then g does not have a section.*

Proof. The case (1) follows from the facts that under the assumptions

$$\widetilde{KO}^{-1}\left(\mathbb{CP}^{2m-1+k'}/\mathbb{CP}^{2m-1}\right) = 0$$

and

$$\widetilde{KO}^{-1}\left(\mathbb{CP}^{2m-1+k'}/\mathbb{CP}^{2m-1+k}\right) = \mathbb{Z}_2.$$

Thus the map f cannot have a section. This proves (1). The proof of (2) also follows from looking at $\widetilde{KO}^{-1}$ of the spaces involved. This completes the proof. □

We remark that ordinary cohomology cannot be used to prove the above non-sectioning result.

Remark 2.5. The main motivation for the computations in Theorem 1.1 was to understand which suspensions $\Sigma^k X_{m,n}$ have the following property: For any vector bundle ξ over $\Sigma^k X_{m,n}$ we have the total Stiefel-Whitney class $w(\xi) = 1$. For example, it follows from our computations that $\widetilde{KO}^{-5}(X_{2m+n,2m-1}) = 0$ if m is even. Thus in this case the suspension $\Sigma^5 X_{2m+n,2m-1}$ has the property that for any vector bundle ξ over $\Sigma^5 X_{2m,2m-1}$ the total Stiefel-Whitney class $w(\xi) = 1$. We have not been able to obtain a complete solution here. A similar question can be asked for suspensions of the stunted quaternionic projective spaces. The computation of the KO-theory of stunted real projective spaces has been used to answer the above question for stunted real projective spaces in [7].

3. Proof of Theorem 1.2

The proof of Theorem 1.2 follows from analyzing the Atiyah-Hirzebruch spectral sequence (AHSS) for KO-theory.

Recall that the AHSS for $\widetilde{KO}(X)$ is given by the spectral sequence with E_2 term given by

$$E_2^{p,q} = \widetilde{H}^p(X; KO^q(\text{point})).$$

It is known that the first non-zero differential

$$d_r^{p,q} : E_r^{p,q} \longrightarrow E_r^{p+r,p-r+1}$$

in the AHSS appears in degree r with $r \equiv 2 \pmod 8$ [6, Proposition 1].

Now let $X = X_{m,n}$, and consider the AHSS for X. Then first note that for $r \geq 2$

$$E_r^{p,q} = 0$$

if $p \not\equiv 0 \pmod 4$. Thus all the differentials in the AHSS (for X) vanish and hence

$$E_2^{p,q} = E_\infty^{p,q}.$$

As X has cells only in even dimension, by Lemma 2.1 of [5], any element of finite order in $\widetilde{KO}^i(X)$ is of order two. This implies that

$$\widetilde{KO}^i(X) = \oplus_{p+q=i} E_\infty^{p,q}.$$

The theorem now follows from the above observations. This completes the proof.

Corollary 3.1. *Let $Y_{m,n}^k = \Sigma^k Y_{m,n}$. Suppose that $k = 0, 4$. If there exists a vector bundle ξ over $Y_{m,n}^k$ with $w(\xi) \neq 1$, then there exists a vector bundle η over $Y_{m',n}^k$ for all $m' > m$ with $w(\eta) \neq 1$.*

Proof. Considering the long exact sequence of $\widetilde{KO}$-groups of the cofiber sequence

$$Y_{m,n} \longrightarrow Y_{m',n} \longrightarrow Y_{m',m},$$

the computations of the $\widetilde{KO}$-groups shows that the homomorphism

$$\widetilde{KO}^{-k}(Y_{m',n}) \longrightarrow \widetilde{KO}^{-k}(Y_{m,n})$$

is surjective. This completes the proof. □

Note that there exists a vector bundle ξ over $Y_{2,1} = S^8$ with $w(\xi) \neq 1$. Thus, by the above corollary, there exists a vector bundle over $X_{m,1}$, $m \geq 2$, with total Stiefel-Whitney class not equal to 1.

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