Doubly commuting submodules of the Hardy module over polydiscs

Jaydeb Sarkar, Amol Sasane and Brett D. Wick

Indian Statistical Institute, Bangalore Centre
8th Mile Mysore Road, Bangalore, 560059 India
Abstract. In this note we establish a vector-valued version of Beurling’s Theorem (the Lax-Halmos Theorem) for the polydisc. As an application of the main result, we provide necessary and sufficient conditions for the completion problem in $H^\infty(D^n)$.

1. Introduction and Statement of Main Results

In [2], Beurling described all the invariant subspaces for the operator $M_z$ of “multiplication by $z$” on the Hilbert space $H^2(D)$ of the disc. In [3], Peter Lax extended Beurling’s result to the (finite-dimensional) vector-valued case (while also considering the Hardy space of the half plane). Lax’s vectorial case proof was further extended to infinite dimensional vector spaces by Halmos, see [8]. The characterization of $M_z$-invariant subspaces obtained is the following famous result.

**Theorem 1.1 (Beurling-Lax-Halmos).** Let $S$ be a closed nonzero subspace of $H^2_{E^*}(D)$. Then $S$ is invariant under multiplication by $z$ if and only if there exists a Hilbert space $E$ and an inner function $\Theta \in H^\infty_{E^*}(D)$ such that $S = \Theta H^2_{E^*}(D)$.

For $n \in \mathbb{N}$ and $E_*$ a Hilbert space, $H^2_{E^*}(D^n)$ is the set of all $E_*$ valued-holomorphic functions in the polydisc $D^n$, where $D := \{z \in \mathbb{C} : |z| < 1\}$ (with boundary $T$) such that

$$\|f\|_{H^2_{E^*}(D^n)} := \sup_{0 < r < 1} \left( \int_{T^n} \|f(rz)\|^2_{E^*} \, dz \right)^{1/2} < +\infty.$$ 

On the other hand, if $\mathcal{L}(E, E_*)$ denotes the set of all continuous linear transformations from $E$ to $E_*$, then $H^\infty_{E^*\to E_*}(D^n)$ denotes the set of all $\mathcal{L}(E, E_*)$-valued holomorphic functions such that $\|f\|_{H^\infty_{E^*\to E_*}(D^n)} := \sup_{z \in D^n} \|f(z)\|_{\mathcal{L}(E, E_*)}$.

A natural question is then to ask what happens in the case of several variables, for example when one considers the Hardy space $H^2_{E^*}(\mathbb{D}^n)$ of the polydisc $\mathbb{D}^n$. It is known that in general, a Beurling-Lax-Halmos type characterization of subspaces of the Hardy Hilbert space is not possible [6]. It is however, easy to see that the Hardy space on the polydisc $H^2_{E^*}(\mathbb{D}^n)$, when $n > 1$, satisfies the *doubly commuting* property, that is, for all $1 \leq i < j \leq n$

$$M^*_i M_j = M_j M^*_i.$$
We impose this additional assumption to the submodules of $H^2_{E,*}(\mathbb{D}^n)$ and call that class of submodules as doubly commuting submodules. More precisely:

**Definition 1.2.** A commuting family of bounded linear operators $\{T_1, \ldots, T_n\}$ on some Hilbert space $\mathcal{H}$ is said to be doubly commuting if

$$T_i T_j^* = T_j^* T_i,$$

for all $1 \leq i, j \leq n$ and $i \neq j$.

A closed subspace $S$ of $H^2_{E,*}(\mathbb{D}^n)$ which is invariant under $M_{z_1}, \ldots, M_{z_n}$ is said to be a doubly commuting submodule if $S$ is a submodule, that is, $M_{z_i} S \subseteq S$ for all $i$ and the family of module multiplication operators $\{R_{z_1}, \ldots, R_{z_n}\}$ where

$$R_{z_i} := M_{z_i}|_S,$$

for all $1 \leq i \leq n$, is doubly commuting, that is,

$$R_{z_i} R_{z_j}^* = R_{z_j}^* R_{z_i},$$

for all $i \neq j$ in $\{1, \ldots, n\}$.

In this note we completely characterize the doubly commuting submodules of the vector-valued Hardy modules $H^2_{E,*}(\mathbb{D}^n)$ over the polydisc, the content of our main theorem. These results are analogues of the classical Beurling-Lax-Halmos Theorem on the Hardy space over the unit disc.

**Theorem 1.3.** Let $S$ be a closed nonzero subspace of $H^2_{E,*}(\mathbb{D}^n)$. Then $S$ is a doubly commuting submodule if and only if there exists a Hilbert space $E$ with $E \subseteq E_*$, where the inclusion is up to unitary equivalence, and an inner function $\Theta \in H^\infty_{E \to E_*}(\mathbb{D}^n)$ such that

$$S = M_{\Theta} H^2_{E,*}(\mathbb{D}^n).$$

In the special scalar case $E_* = \mathbb{C}$ and when $n = 2$ (the bidisc), this characterization is known, see for example [4], and the proof given there relies on the Wold decomposition [7]. In this note, using the more natural language of Hilbert modules, we give a new proof of this characterization that avoids appealing to the Wold decomposition, and additionally, works for all $n$ simultaneously.

As an application of this theorem, we can establish a version of the Completion Property for the algebra $H^\infty(\mathbb{D}^n)$. Suppose that $E \subset E_*$. Recall that the Completion Problem for $H^\infty(\mathbb{D}^n)$ is the problem of characterizing the functions $f \in H^\infty_{E \to E_*}(\mathbb{D}^n)$ such that there exists an invertible function $F \in H^\infty_{E_* \to E_*}(\mathbb{D}^n)$ with $F|_E = f$.

In the case of $H^\infty(\mathbb{D})$, the Completion Problem was settled by Tolokonnikov in [9]. In that paper, it was pointed out that there is a close connection between the Completion Problem and the characterization of invariant subspaces of $H^2(\mathbb{D})$. Using Theorem 1.3 we then have the following analogue of the results in [9].

**Theorem 1.4 (Tolokonnikov’s Lemma for the Polydisc).** Let $f \in H^\infty_{E \to E_*}(\mathbb{D}^n)$ with $E \subset E_*$ and $\dim E < \infty$. Then the following statements are equivalent:

(i) There exists a function $g \in H^\infty_{E_* \to E}(\mathbb{D}^n)$ such that $gf \equiv I$ in $\mathbb{D}^n$ and the operators $M_{z_1}, \ldots, M_{z_n}$ doubly commute on the subspace $\ker M_g$. 


(ii) There exists a function $F \in H_{E_c \rightarrow E_c}^{\infty}(\mathbb{D}^n)$ such that $F|_E = f$, $F|_{E_c \ominus E}$ is inner, and $F^{-1} \in H_{E_c \rightarrow E_c}^{\infty}(\mathbb{D}^n)$.

In Section 2 we give a proof of Theorem 1.3, and subsequently, in Section 3, we use this theorem to study the Completion Problem for $H^\infty(\mathbb{D}^n)$, providing a proof of Theorem 1.4.

2. BEURLING-LAX-HALMOS THEOREM FOR THE POLYDISC

We now turn to the proof of Theorem 1.3. But, first we give a characterization of “reducing submodules” of $H^2_E(\mathbb{D}^n)$ since it will be useful in our work.

**Definition 2.1.** A closed subspace $S \subseteq H^2_E(\mathbb{D}^n)$ is said to be a reducing submodule of $H^2_E(\mathbb{D}^n)$ if $M_{z_i}S, M_{z_i}^*S \subseteq S$ for all $i = 1, \ldots, n$.

**Lemma 2.2.** Let $S$ be a closed subspace of $H^2_E(\mathbb{D}^n)$. Then $S$ is a reducing submodule of $H^2_E(\mathbb{D}^n)$ if and only if

$$S = H^2_{E_*(\mathbb{D}^n)},$$

for some closed subspace $E_*$ of $E$.

**Proof.** Let $S$ be a reducing submodule of $H^2_E(\mathbb{D}^n)$, that is, for all $1 \leq i \leq n$ we have

$$M_{z_i}P_S = P_SM_{z_i}.$$

Let

$$S(z, w) = \prod_{j=1}^n (1 - \bar{w}_j z_j)^{-1},$$

be the Cauchy kernel on the polydisc $\mathbb{D}^n$. Now following Agler’s hereditary functional calculus [1], we have

$$S^{-1}(M_z, M_z) = \left( \prod_{i=1}^n (1 - z_i \bar{w}_i) \right) (M_z, M_z)
= \sum_{0 \leq i_1 < \ldots < i_t \leq n} (-1)^t (z_{i_1} \ldots z_{i_t} \bar{w}_{i_1} \ldots \bar{w}_{i_t}) (M_z, M_z)
= \sum_{0 \leq i_1 < \ldots < i_t \leq n} (-1)^t M_{z_{i_1}} \ldots M_{z_{i_t}} M^*_{z_{i_1}} \ldots M^*_{z_{i_t}}.$$
and hence for all $z, w \in \mathbb{D}^n$ and $\eta, \zeta \in E$ we have

$$\langle S^{-1}(M_z, M_z) S(\cdot, z) \eta, S(\cdot, w) \zeta \rangle$$

$$= \left\langle \sum_{0 \leq i_1 < \cdots < i_n \leq n} (-1)^i M_{z_{i_1}} \cdots M_{z_{i_n}} \cdot M^* \cdot S(\cdot, z) \eta, S(\cdot, w) \zeta \right\rangle$$

$$= \sum_{0 \leq i_1 < \cdots < i_n \leq n} (-1)^i \left\langle M^*_z \cdot M^* \cdot S(\cdot, z) \eta, M^* \cdot M^*_z \cdot S(\cdot, w) \zeta \right\rangle$$

$$= \sum_{0 \leq i_1 < \cdots < i_n \leq n} (-1)^i \langle z_{i_1} \cdots z_{i_n} w_{i_1} \cdots w_{i_n} \langle S(\cdot, z), S(\cdot, w) \rangle \langle \eta, \zeta \rangle$$

$$= S^{-1}(w, z) S(w, z) \langle \eta, \zeta \rangle$$

$$= \langle \eta, \zeta \rangle$$

$$= \langle P_E S(\cdot, z) \eta, S(\cdot, w) \zeta \rangle$$

where $P_E$ denotes the orthogonal projection of $H^2_E(\mathbb{D}^n)$ onto the space of all constant functions. Since $\{S(\cdot, z) \eta : z \in \mathbb{D}^n, \eta \in E\}$ is a total subset of $H^2_E(\mathbb{D}^n)$, we have that

$$S^{-1}(M_z, M_z) = P_E.$$

Consequently,

$$P_E P_S = S^{-1}(M_z, M_z) P_S = P_S S^{-1}(M_z, M_z) = P_S P_E.$$ 

Therefore, $P_S P_E$ is an orthogonal projection and

$$P_S P_E = P_E P_S = P_{E*},$$

where $E* := E \cap S$. Hence, for any

$$f = \sum_{k \in \mathbb{N}^n} a_k z^k \in S,$$

where $a_k \in E$ for all $k \in \mathbb{N}^n$, we have

$$f = P_S f = P_S \left( \sum_{k \in \mathbb{N}^n} M^*_z a_k \right) = \sum_{k \in \mathbb{N}^n} M^*_z P_S a_k.$$

But $P_S a_k = P_S P_E a_k \in E*$. Consequently, $M^*_z P_S a_k \in H^2_{E*}(\mathbb{D}^n)$ for all $k \in \mathbb{N}^n$ and hence $f \in H^2_{E*}(\mathbb{D}^n)$. That is, $S \subseteq H^2_{E*}(\mathbb{D}^n)$. For the reverse inclusion, it is enough to observe that $E* \subseteq S$ and that $S$ is a reducing submodule. The converse part is immediate. Hence the lemma follows.

With this lemma, we can now give the proof of Theorem 1.3

**Proof of Theorem 1.3.** Let $S$ be a doubly commuting submodule of $H^2_{E*}(\mathbb{D}^n)$ and

$$\mathcal{R} = \{f \in S : M^*_z f \in S, \forall k \in \mathbb{N}^n\}.$$ 

Clearly, $\mathcal{R} \subseteq S$ is a reducing submodule of $H^2_{E*}(\mathbb{D}^n)$. Therefore,

$$S = \mathcal{R} \oplus (S \ominus \mathcal{R}),$$
and hence, Lemma 2.2 implies that

\[ S = H_{E_1}^2 \oplus \tilde{S}, \]

for some closed subspace \( E_1 \subseteq E_\ast \) and that \( \tilde{S} := S \ominus R \) is a doubly commuting submodule containing no non-zero reducing submodule of \( H_{E_1}^2(\mathbb{D}^n) \). Then by the double commutativity of \( \tilde{S} \)

\[ (P_{\tilde{S}} - R_{z_i}^* R_{z_j}^*)(P_{\tilde{S}} - R_{z_j}^* R_{z_j}^*) = (P_{\tilde{S}} - R_{z_j}^* R_{z_j}^*)(P_{\tilde{S}} - R_{z_i}^* R_{z_i}^*), \]

for all \( i \neq j \). Also

\[ S^{-1}(R, R) := \sum_{0 \leq i_1 < \ldots < i_l \leq n} (-1)^l R_{z_{i_1}} \ldots R_{z_{i_l}} R_{z_{i_1}}^* \ldots R_{z_{i_l}}^* \]

implies that \( R_{z_i} R_{z_i}^* \) is an orthogonal projection of \( S \) onto \( z_i S \) and hence \( P_{\tilde{S}} - R_{z_i} R_{z_i}^* \) is an orthogonal projection of \( S \) onto \( S \ominus z_i S \), that is,

\[ P_{\tilde{S}} - R_{z_i} R_{z_i}^* = P_{S \ominus z_i S}, \]

for all \( i = 1, \ldots, n \). Therefore, \( S^{-1}(R, R) \) is the product of commuting orthogonal projections \( P_{S \ominus z_i S} \), that is,

\[ S^{-1}(R, R) = \prod_{i=1}^{n} (P_{\tilde{S}} - R_{z_i} R_{z_i}^*) = \prod_{i=1}^{n} P_{S \ominus z_i S} = P_{E_2}, \]

where \( E_2 := \cap_{i=1}^{n} (S \ominus z_i \tilde{S}) \). Now define \( V : H_{E_2}^2(\mathbb{D}^n) \rightarrow H_{E_4}^2(\mathbb{D}^n) \) by

\[ V \left( \sum_{k \in \mathbb{N}^n} a_k z^k \right) = \sum_{k \in \mathbb{N}^n} M_z^k a_k, \]

where \( \sum_{k \in \mathbb{N}^n} a_k z^k \in H_{E_2}^2(\mathbb{D}^n) \) and \( a_k \in E_2 \) for all \( k \in \mathbb{N}^n \). It is evident that \( V \in \mathcal{L}(H_{E_2}^2(\mathbb{D}^n), H_{E_4}^2(\mathbb{D}^n)) \) is isometric module map. Therefore,

\[ V = M_{\Theta_2}, \]

for some inner function \( \Theta_2 \in H_{E_2 \rightarrow E_4}(\mathbb{D}^n) \). Moreover, since \( \tilde{S} \) is a submodule and \( E_2 \subseteq \tilde{S} \), we have

\[ \text{ran} V = \text{ran} M_{\Theta_2} \subseteq \tilde{S}. \]

We claim that \( \text{ran} V = \tilde{S} \). If not, let \( f \in \tilde{S} \) and \( f \perp \text{ran} V \), or equivalently, for all \( k \in \mathbb{N}^n \)

\[ M_z^k f \perp \left( E_2 = \bigcap_{i=1}^{n} (S \ominus z_i S) = \tilde{S} \ominus \left( \sum_{i=1}^{n} z_i \tilde{S} \right) \right). \]
Then for all $k \in \mathbb{N}^n$, 
\[ M^kf \in \sum_{i=1}^n z_i \tilde{S} \subseteq \tilde{S}, \]
and hence 
\[ \text{span}\{M^kf : k \in \mathbb{N}^n\} \subseteq \tilde{S} \]
is a reducing submodule of $H^2_{E_1}(\mathbb{D}^n)$ which is contained in $\tilde{S}$, contradicting the fact that $\tilde{S}$ is irreducible. Consequently, 
\[ f = 0, \]
and $\tilde{S} = \text{ran} V = M_{\Theta} H^2_{E_2}(\mathbb{D}^n)$. Finally, we define 
\[ W : H^2_{E_1}(\mathbb{D}^n) \oplus H^2_{E_2}(\mathbb{D}^n) \rightarrow H^2_{E_1}(\mathbb{D}^n), \]
by 
\[ W(f_1 \oplus f_2) = f_1 \oplus Vf_2 = f_1 \oplus M_{\Theta} f_2, \]
for all $f_i \in H^2_{E_i}(\mathbb{D}^n)$ and $i = 1, 2$. Then $W$ is a module isometric map from $H^2_{E_1 \oplus E_2}(\mathbb{D}^n) = H^2_{E_1}(\mathbb{D}^n) \oplus H^2_{E_2}(\mathbb{D}^n)$ to $H^2_{E_1}(\mathbb{D}^n)$. Defining $E := E_1 \oplus E_2$ we have that $W = M_{\Theta}$ for some inner function $\Theta \in H^\infty_{E \rightarrow E_1}(\mathbb{D}^n)$ and $\tilde{S} = M_{\Theta} H^2_{E_1}(\mathbb{D}^n)$.

To prove the converse part, let $S = M_{\Theta} H^2_{E}(\mathbb{D}^n)$ be a submodule of $H^2_{E_1}(\mathbb{D}^n)$ for some inner function $\Theta \in H^\infty_{E \rightarrow E_1}(\mathbb{D}^n)$. Then 
\[ P_{S} = M_{\Theta} M^*_{\Theta}, \]
and hence for all $i \neq j$,
\[ M_z P_{S} M^*_z = M_z M_{\Theta} M^*_{\Theta} M^*_z = M_{\Theta} M_z M^*_z M^*_{\Theta} = M_{\Theta} M^*_z M^*_{\Theta} M_z M^*_z, \]
which implies 
\[ R^*_z R_z = P_{S} M^*_z P_{S} M_z |_{S} = P_{S} M^*_z M_z |_{S} = M_z P_{S} M^*_z = R_z R^*_z, \]
that is, $S$ is a doubly commuting submodule. This completes the proof. \hfill \blacksquare

3. Tolokonnikov’s Lemma for the Polydisc

We will need the following lemma, which is a polydisc version of a similar result proved in the case of the disc in Nikolski’s book [5, p.44-45]. Here we use the notation $M_g$ for the multiplication operator on $H^2_E$ induced by $g \in H^\infty_{E \rightarrow E_1}$.

Lemma 3.1 (Lemma on Local Rank). Let $E, E_c$ be Hilbert spaces, with $\dim E < \infty$. Let 
\[ g \in H^\infty_{E_1 \rightarrow E}(\mathbb{D}^n) \]
be such that 
\[ \ker M_g = \{ h \in H^2_{E_1}(\mathbb{D}^n) : g(z)h(z) \equiv 0 \} = \Theta H^2_{E_1}(\mathbb{D}^n), \]
where $E_a$ is a Hilbert space and $\Theta$ is a $\mathcal{L}(E_a, E_c)$-valued inner function. Then 
\[ \dim E_c = \dim E_a + \text{rank } g, \]
where $\text{rank } g := \max_{\zeta \in \mathbb{D}^n} \text{rank } g(\zeta)$. 

Proof. We have \( \ker M_g = \{ h \in H^2_{E_c}(\mathbb{D}^n) : gh \equiv 0 \} \). If \( \zeta \in \mathbb{D}^n \), then let 
\[
[\ker M_g](\zeta) := \{ h(\zeta) : h \in \ker M_g \}.
\]
It is easy to check that \( [\ker M_g](\zeta) = \Theta(\zeta)E_a \). If \( \dim E_c = \infty \), then one can show that \( \dim[\ker M_g](\zeta) = \infty \). So \( \dim E_a = \infty \) as well, and this proves the claim.

So we assume that \( \dim E_c < \infty \). It is clear that for \( \zeta \in \mathbb{D}^n \),
\[
\dim \Theta(\zeta)E_a = \dim[\ker M_g](\zeta) \leq \dim \ker g(\zeta) = \dim E_c - \operatorname{rank} g(\zeta).
\]
From the analyticity of \( \Theta \) and \( g \), it follows that there exists a point \( \zeta_1 \in \mathbb{D}^n \), with
\[
\dim E_a = \dim \Theta(\zeta_1)E_a, \quad \operatorname{rank} g(\zeta_1) = \operatorname{rank} g.
\]
Hence \( \dim E_a \leq \dim E_c - \operatorname{rank} g \).

For the proof of the opposite inequality, let us consider a principal minor \( g_1(\zeta_1) \) of the matrix of the operator \( g(\zeta_1) \) (with respect to two arbitrary fixed bases in \( E_c \) and \( E \) respectively). Then \( \det g_1 \in H^\infty \), \( \det g_1 \neq 0 \). Let \( E_c = E_{c,1} \oplus E_{c,2} \), \( E = E_1 \oplus E_2 \) (\( \dim E_{c,1} = \dim E_1 = \operatorname{rank} g(\zeta_1) \)) be the decompositions of the spaces \( E_c \) and \( E \) corresponding to this minor, and let
\[
g(\zeta) = \begin{bmatrix}
g_1(\zeta) & g_2(\zeta) \\
g_1(\zeta) & g_2(\zeta)
\end{bmatrix}, \quad \zeta \in \mathbb{D}^n,
\]
be the matrix representation of \( g(\zeta) \) with respect to this decomposition. Using
\[
\gamma_2 \det g_1 = \gamma_1 g_1^{co} g_2, \quad \text{where} \quad g_1^{co} := (\det g_1)^{-1},
\]
we get the inclusion \( M_{\Omega} H^2_{E_{c,2}}(\mathbb{D}^n) \subset \ker M_g \), where \( \Omega \in H^\infty_{E_{c,2} \rightarrow E_c}(\mathbb{D}^n) \) is given by
\[
\Omega = \begin{bmatrix}
g_1^{co} g_2 \\
- \det g_1
\end{bmatrix}.
\]
We have \( \operatorname{rank} \Omega = \dim E_{c,2} = \dim E_c - \operatorname{rank} g = \dim \ker(g(\zeta_1)) \). Consequently, we obtain
\[
\dim[\ker M_g](\zeta_1) \geq \dim \ker(g(\zeta_1)).
\]
We now turn to the extension of Tolokonnikov’s Lemma to the polydisc.

Proof of Theorem 1.4. (ii) \( \Rightarrow \) (i): If \( g := P_E F^{-1} \), then \( gf = I \). It only remains to show that the operators \( M_{z_1}, \ldots, M_{z_n} \) are doubly commuting on the space ker \( M_g \). Let \( \Theta, \Gamma \) be such that:
\[
F = \begin{bmatrix}
f & \Theta \\
\end{bmatrix} \quad \text{and} \quad F^{-1} = \begin{bmatrix}
g & \Gamma
\end{bmatrix}.
\]
Since \( FF^{-1} = I_{E_c} \), it follows that \( f \Theta + \Theta \Gamma = I_{E_c} \). Thus if \( h \in H^2_{E_c}(\mathbb{D}^n) \) is such that \( gh = 0 \), then \( \Theta(\Gamma h) = h \), and so \( h \in \Theta H^2_{E_c \oplus E}(\mathbb{D}^n) \). Hence ker \( M_g \subset \text{ran } M_\Theta \). Also, since \( F^{-1} F = I \), it follows that \( g \Theta = 0 \), and so \( \text{ran } M_{\Theta} \subset \ker M_g \). Consequently, ker \( M_g = \text{ran } M_{\Theta} = \Theta H^2_{E_c \oplus E}(\mathbb{D}^2) \). By Theorem 1.3, the operators \( M_{z_1}, \ldots, M_{z_n} \) must doubly commute on the subspace ker \( M_g \).

(i) \( \Rightarrow \) (ii): Let
\[
S := \{ h \in H^2_{E_c}(\mathbb{D}^n) : g(z)h(z) \equiv 0 \} = \ker g.
\]
\( S \) is a closed non-zero invariant subspace of \( H^2_{E_c}(\mathbb{D}^n) \). Also, by assumption, \( M_{z_1}, \ldots, M_{z_n} \) are doubly commuting operators on \( S \). Then by the above Theorem 1.3, there exists an auxiliary
Hilbert space $E_a$ and an inner function $\tilde{\Theta}$ with values in $L(E_a, E_c)$ with $\dim E_a \leq \dim E_c$ such that

$$S = \tilde{\Theta}H^2_{E_c}(\mathbb{D}^n).$$

Note that by the Lemma on Local Rank above, $\dim E_a = \dim E_c - \text{rank } g = \dim E_c - \dim E = \dim (E_c \oplus E)$. Let $U$ be a (constant) unitary operator from $E_c \oplus E$ to $E_a$ and define $\Theta := \tilde{\Theta}U$. Then $\Theta$ is inner, and we have that $\ker g = \Theta H^2_{E_c \oplus E}(\mathbb{D}^n)$. To get $F \in H^\infty_{E_c \to E_c}(\mathbb{D}^n)$ define the function $F$ for $z \in \mathbb{D}^n$ by

$$F(z)e := \begin{cases} f(z)e & \text{if } e \in E \\ \Theta(z)e & \text{if } e \in E_c \oplus E. \end{cases}$$

We note that $F \in H^\infty(\mathbb{D}^n)$ and $F|_E = f$. We now show that $F$ is invertible. With this in mind, we first observe that

$$(I - fg)\Theta H^2_{E_c \oplus E}(\mathbb{D}^n) \subset \Theta H^2_{E_c \oplus E}(\mathbb{D}^n) = \ker M_g.$$ 

This follows since $g(I - fg)h = gh - gh = 0$ for all $h \in H^2_{E_c}(\mathbb{D}^n)$. Thus, $\Theta^*(I - fg) \in H^\infty_{E_c \to E_c \oplus E}(\mathbb{D}^n)$. Now, define $\Omega = g \oplus \Theta^*(I - fg)$, and we clearly have $\Omega \in H^\infty_{E_c \to E_c}(\mathbb{D}^n)$. Next, note that

$$F\Omega = fg + \Theta\Theta^*(I - fg) = I.$$ 

Similarly,

$$\Omega F = gfP_E + \Theta^*(I - fg)(fP_E + \Theta P_{E_c \oplus E}) = P_E + \Theta^*(fP_E - ffgP_E + \Theta P_{E_c \oplus E}) = P_E + \Theta^*\Theta P_{E_c \oplus E} = I.$$ 

Thus we have that $F^{-1} \in H^\infty(\mathbb{D}^n; E_c \to E_c)$. \hfill \qed

**Remark 3.2.** Theorem 1.4 for the polydisc is different from Tolokonnikov’s lemma in the disc in which one does not demand that the completion $F$ has the property that $F|_{E_c \oplus E}$ is inner. But, from the proof of Tolokonnikov’s lemma in the case of the disc (see [5]), one can see that the following statements are equivalent for $f \in H^\infty_{E_c \to E_c}(\mathbb{D})$ with $E \subset E_c$ and $\dim E < \infty$:

(i) There exists a function $g \in H^\infty_{E_c \to E_c}(\mathbb{D})$ such that $gf \equiv 1$ in $\mathbb{D}^n$.

(ii) There exists a function $F \in H^\infty_{E_c \to E_c}(\mathbb{D})$ such that $F|_E = f$, and $F^{-1} \in H^\infty_{E_c \to E_c}(\mathbb{D})$.

(iii) There exists a function $F \in H^\infty_{E_c \to E_c}(\mathbb{D})$ such that $F|_E = f$, $F|_{E_c \oplus E}$ is inner, and $F^{-1} \in H^\infty_{E_c \to E_c}(\mathbb{D})$.

**References**


J. Sarkar, Indian Statistical Institute, Statistics and Mathematics Unit, 8th Mile, Mysore Road, Bangalore, 560059, India

E-mail address: jay@isibang.ac.in, jaydeb@gmail.com

URL: http://www.isibang.ac.in/~jay/

A. Sasane, Mathematics Department, London School of Economics, Houghton Street, London WC2A 2AE, U.K.

E-mail address: sasane@lse.ac.uk

Brett D. Wick, School of Mathematics, Georgia Institute of Technology, 686 Cherry Street, Atlanta, GA USA 30332-0160, U.S.A.

E-mail address: wick@math.gatech.edu

URL: www.math.gatech.edu/~wick